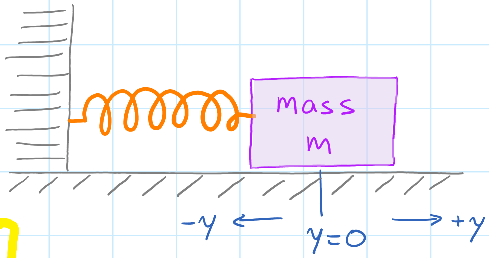
UNDAMPED SPRING-MASS SYSTEM

Consider a mass attached to a spring, sliding horizontally on a frictionless surface.

Let $y(t)$ be the position of the mass at time t.



Newton's 2nd Law: $F = ma$, so $\boxed{F = m\frac{d^2y}{dt^2}}$

Hooke's Law of Springs: restoring force is proportional to displacement

$\boxed{F = -ky}$, where $k>0$ is the spring constant

Thus: $m\frac{d^2y}{dt^2} = -ky$ or $\boxed{\frac{d^2y}{dt^2} + \frac{k}{m}y = 0}$ second-order eq. for the motion of a simple harmonic oscillator

Convert eq. to a system of eq:

Let $\underbrace{v(t) = \frac{dy}{dt}}$, the velocity of the mass at time t

Then $\frac{dv}{dt} = \frac{d^2y}{dt^2}$, so the equation $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$ becomes:

$$\boxed{\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m}y \end{cases}}$$

System of two first-order equations

EXAMPLE: Suppose a mass weighing 10 pounds stretches a spring 2 inches. $= \frac{1}{6}$ ft.

The spring constant is thus $k = \frac{10 \text{ pounds}}{\frac{1}{6} \text{ inches}} = 60$ pounds/foot

Since $F = mg$, the mass is $m = \frac{F}{g} = \frac{10 \text{ pounds}}{32 \text{ ft/sec}^2} = \frac{5}{16}$

Equations: $\frac{d^2y}{dt^2} + \frac{60}{\frac{5}{16}}y = 0$

$$\boxed{\frac{d^2y}{dt^2} + 192y = 0} \quad \text{or} \quad \boxed{\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -192y \end{cases}}$$

VISUALIZING SOLUTIONS:

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Let $\vec{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$. This vector field gives solution curves in the yv-plane (the phase plane).

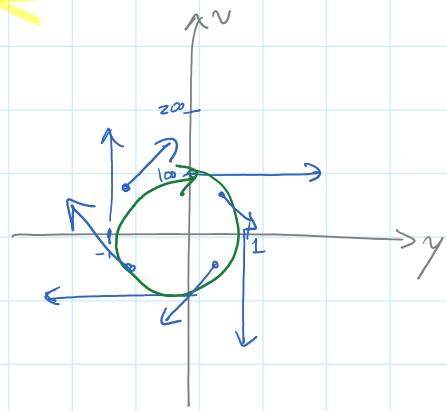
Let $\vec{F}(t) = \dfrac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{bmatrix}$. This gives tangent vectors to the solution curves.

For the previous example: $\vec{F}(t) = \begin{bmatrix} v \\ -192y \end{bmatrix}$

If $y=1$, $v=0$, then vector is $\begin{bmatrix} 0 \\ -192 \end{bmatrix}$

If $y=0$, $v=100$ then $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$

If $y=1$, $v=1$, then $\begin{bmatrix} 1 \\ -192 \end{bmatrix}$



WORKSHEET SOLUTIONS:

1. $k = \dfrac{200\,N}{5\,cm} = 40\ N/cm$

2. $m = 2\,kg$

(a) $\dfrac{d^2y}{dt^2} + \dfrac{40}{2}y = 0$

(b) $\begin{cases} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = -20y \end{cases}$

(c) initial position: $y(0) = 10$ cm

initial velocity: $v(0) = 0$

3. Is $y(t) = \sin(\beta t)$ a solution?

$\dfrac{d^2y}{dt^2} = -\beta^2 \sin(\beta t)$

Substitute in to $\dfrac{d^2y}{dt^2} + 20y = 0$

$$-\beta^2 \sin(\beta t) + 20\sin(\beta t) = 0$$

$$\underbrace{\sin(\beta t)}\underbrace{\left(-\beta^2 + 20\right)} = 0$$

If $\sin(\beta t) = 0$, then $\beta = 0$, which gives the trivial solution $y(t) = 0$.

If $-\beta^2 + 20 = 0$, then $\beta^2 = 20$, so $\beta = \pm\sqrt{20}$.

Then $y(t) = \pm \sin\left(\sqrt{20}\, t\right)$ is a solution.

Also: $y(t) = \cos(\beta t)$ is a solution if $\beta = \pm\sqrt{20}$.

General solution to $\frac{d^2y}{dt^2} + 20y = 0$ is:

$$\boxed{y(t) = A\sin\left(\sqrt{20}\, t\right) + B\cos\left(\sqrt{20}\, t\right) \qquad A, B \in \mathbb{R}}$$

So: $v(t) = \sqrt{20} A \cos\left(\sqrt{20}\, t\right) - \sqrt{20} B \sin\left(\sqrt{20}\, t\right)$

4. Phase portrait: