LAST TIME: $\boxed{\dfrac{d^2y}{dt^2} + \dfrac{k}{m}y = 0}$

k: spring constant
m: mass

solutions: sin( ) and cos( ) functions

Why are these the solutions?

It must be that the second derivative $\frac{d^2y}{dt^2}$ is $-\frac{k}{m}$ times the function $y(t)$

$$\frac{d^2y}{dt^2} = \frac{-k}{m}y$$

What functions have this property?

$y_1(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$y_1'(t) = \sqrt{\frac{k}{m}}A\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

$y_1''(t) - \frac{k}{m}A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$y_2(t) = B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

$y_2'(t) = -B\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$y_2''(t) = -B\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

General Solution: $y(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

## NOW ADD FRICTION TO THE SPRING-MASS SYSTEM

Assume: friction (damping) force is proportional to velocity, but in the opposite direction.

That is: $F_d = -b\dfrac{dy}{dt}$ for some constant $b > 0$

Then: $\underbrace{m\dfrac{d^2y}{dt^2}}_{\substack{\text{force} \\ F=ma}} = \underbrace{-b\dfrac{dy}{dt}}_{\substack{\text{damping} \\ \text{force}}} \underbrace{-ky}_{\substack{\text{restoring} \\ \text{force}}}$ or $\dfrac{d^2y}{dt^2} + \dfrac{b}{m}\dfrac{dy}{dt} + \dfrac{k}{m}y = 0$

Let $p = \frac{b}{m}$, $q = \frac{k}{m}$, so the equation becomes $\boxed{\dfrac{d^2y}{dt^2} + p\dfrac{dy}{dt} + qy = 0}$

$p \geq 0$, $q > 0$

EXAMPLE: $\dfrac{d^2y}{dt^2} + 9\dfrac{dy}{dt} + 14y = 0$ What are the solutions?

Try: $y(t) = e^{st}$ for some $s \in \mathbb{R}$

$y'(t) = se^{st}$

$y''(t) = s^2e^{st}$

plug in to $y'' + 9y' + 14y = 0$

$s^2e^{st} + 9se^{st} + 14e^{st} = 0$

$\cancel{e^{st}}(s^2 + 9s + 14) = 0$

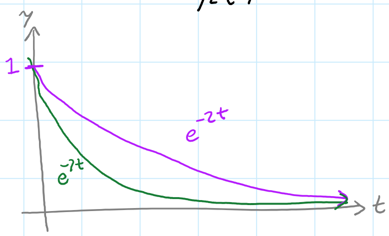
$(s+2)(s+7) = 0$

Solutions: $y_1(t) = e^{-2t}$

$$\cancel{e^{st}}\,(s^2 + 9s + 14) = 0$$
$$(s+2)(s+7) = 0$$

Solutions: $y_1(t) = e^{-2t}$

$y_2(t) = e^{-7t}$ $\longleftarrow$ so $s = -2$ or $-7$




General solution:

$$y(t) = A e^{-2t} + B e^{-7t}$$

## EXAMPLE: $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 9y = 0$

Find two solutions that are not constant multiples of each other.

Try: $y(t) = e^{st}$ :

$$y'' + 6y' + 9y = s^2 e^{st} + 6se^{st} + 9e^{st} = 0$$
$$e^{st}(s^2 + 6s + 9) = 0$$
$$(s+3)^2 = 0$$
$$s = -3$$

solution: $y_1(t) = e^{-3t}$

Now try: $y(t) = t e^{st}$

"test" function!

$$y'(t) = 1e^{st} + tse^{st} = (1+st)e^{st}$$
$$y''(t) = (s)e^{st} + (1+st)se^{st} = (2s + s^2t)e^{st}$$

Plug in: $y'' + 6y' + 9y = (2s + s^2t)e^{st} + 6(1+st)e^{st} + 9te^{st} = 0$

$$\cancel{e^{st}}\,(2s + s^2t + 6 + 6st + 9t) = 0$$
$$s^2t + 6st + 9t + 2s + 6 = 0$$
$$\underbrace{(s^2 + 6s + 9)t}_{\text{must be } 0} + \underbrace{(2s+6)}_{\text{must be } 0} = 0$$

$s^2 + 6s + 9 = 0$

$(s+3)^2 = 0$

$s = -3$

$2s + 6 = 0$

$s = -3$

Solution:

$y_2(t) = t e^{-3t}$

General solution: $y(t) = Ae^{-3t} + Bte^{-3t}$

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In[2]:= Plot[{Exp[-3 t], t * Exp[-3 t]}, {t, 0, 5}, PlotRange → {0, 1}]
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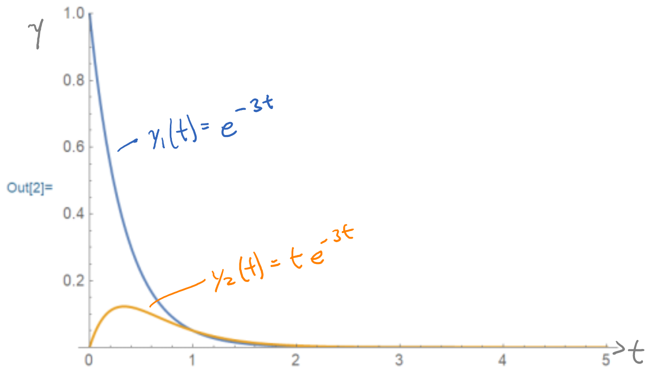
Out[2]=




If I have initial conditions $y(0)=5$, $y'(0)=-2$:

$$y(0) = Ae^0 + B(0)e^0 = A = 5$$

$$y'(t) = -3Ae^{-3t} + Be^{-3t} - 3Bte^{-3t}$$

so $y'(0) = -3Ae^0 + Be^0 - 3(0)e^0 = -3A + B = -2$

$$-3(5) + B = -2$$

$$B = -2 + 15 = 13$$

Particular Solution: $y(t) = 5e^{-3t} + 13te^{-3t}$

Go back to the diff. eq: $\overset{\frac{dv}{dt}}{\frac{d^2y}{dt^2}} + 6\overset{v}{\left(\frac{dy}{dt}\right)} + 9y = 0$

Let $v(t) = \frac{dy}{dt}$, so $\frac{dv}{dt} = \frac{d^2y}{dt}$. The equation becomes:

$$\boxed{\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -6v - 9y \end{cases}}$$

plot the phase portrait

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In[3]:= StreamPlot[{v, -6 v - 9 y}, {y, -5, 5},
        {v, -5, 5}]
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Out[3]=