CHAPTER 3:

We want to solve the linear system

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

for all choices of constants a, b, c, d

We write the system as $\frac{d\vec{Y}}{dt} = A\vec{Y}$, where $\vec{Y} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If we solve the system above, then we can also solve

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \rightarrow \begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -qy - pv \end{cases}$$

TRIVIAL SOLUTION:

$\vec{Y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a equilibrium solution to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ for all 2x2 matrices A.

why? $\frac{d\vec{Y}_0}{dt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1. OTHER EQUILIBRIUM SOLUTIONS?

If $\vec{Y} = \begin{bmatrix} r \\ s \end{bmatrix}$ solves $\frac{d\vec{Y}}{dt} = A\vec{Y}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

constants

then $\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So: $\underset{\text{zero}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\text{nonzero}}{\begin{bmatrix} r \\ s \end{bmatrix}}$

This only happens if $\det(A) = 0$.

If $\det(A) \neq 0$, then $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has only one equilibrium solution, $\vec{Y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

If $\det(A) = 0$, then $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has nontrivial equilibrium solutions.

2. For which matrices A does $\frac{d\vec{Y}}{dt} = A\vec{Y}$ have a nontrivial equilibrium solution?

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$	$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$	$A = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$	$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$
$\det(A) = 1 - 4 = -3$	$\det(A) = 0$	$\det(A) = 0$	$\det(A) = 4$
No.	Yes.	Yes.	No.

Nontrivial eq. solutions

3. If $\vec{Y}(t) = \begin{bmatrix} x(t) \\ \dots \end{bmatrix}$ is a solution of $\frac{d\vec{Y}}{dt} = A\vec{Y}$, then $k\vec{Y}$ is also a

3. If $\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of $\frac{d\vec{Y}}{dt} = A\vec{Y}$, then $k\vec{Y}$ is also a solution for any constant k.

We know: $\frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$\frac{d\vec{Y}}{dt} = A\vec{Y}$

show $k\vec{Y}$ is a solution:

$\frac{d}{dt}(k\vec{Y}) = k\frac{d\vec{Y}}{dt} = k(A\vec{Y}) = A(k\vec{Y})$

Thus: $\frac{d}{dt}(k\vec{Y}) = A(k\vec{Y})$

Conclude: $k\vec{Y}$ is a solution.

4. If $\vec{Y}_1$ and $\vec{Y}_2$ are solutions, then so is $\vec{Y}_1 + \vec{Y}_2$.

we know: $\frac{d\vec{Y}_1}{dt} = A\vec{Y}_1$ and $\frac{d\vec{Y}_2}{dt} = A\vec{Y}_2$.

show:

$\frac{d}{dt}(\vec{Y}_1 + \vec{Y}_2) = \frac{d\vec{Y}_1}{dt} + \frac{d\vec{Y}_2}{dt} = A\vec{Y}_1 + A\vec{Y}_2 = A(\vec{Y}_1 + \vec{Y}_2)$

Thus: $\frac{d}{dt}(\vec{Y}_1 + \vec{Y}_2) = A(\vec{Y}_1 + \vec{Y}_2)$

So $\vec{Y}_1 + \vec{Y}_2$ is a solution.

5. $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, $\vec{Y}_1 = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$, $\vec{Y}_2 = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$

(a) Verify that $\vec{Y}_1$ and $\vec{Y}_2$ solve $\frac{d\vec{Y}}{dt} = A\vec{Y}$

deriv:

$$\frac{d\vec{Y}_1}{dt} = \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix} \stackrel{?}{=} A\vec{Y}_1 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} - 2e^{-t} \\ 4e^{-t} - 2e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ 2e^{-t} \end{bmatrix}$$

$$\frac{d\vec{Y}_2}{dt} = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix} \stackrel{?}{=} A\vec{Y}_2 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} e^{3t} + 2e^{3t} \\ 4e^{3t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 6e^{3t} \end{bmatrix}$$

(b) Find a solution to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with initial condition $\vec{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{Y}_1(0) = \begin{bmatrix} e^0 \\ -2e^0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad$ $\vec{Y}_2(0) = \begin{bmatrix} e^0 \\ 2e^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Find k_1 and k_2 such that $k_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solve: $\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

row reduce: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \end{array}\right]$ So: $k_1 = -\frac{1}{4}$, $k_2 = \frac{1}{4}$

Thus: $\vec{Y} = -\frac{1}{4}\vec{Y}_1 + \frac{1}{4}\vec{Y}_2$ is the solution to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with $\vec{Y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{Y}(t) = \begin{bmatrix} -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} \\ \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{bmatrix}$$