SECOND-ORDER LINEAR EQUATIONS

$$\boxed{a\underbrace{\frac{d^2y}{dt^2}}_{\frac{dv}{dt}} + b\underbrace{\frac{dy}{dt}}_{v} + cy = 0} \qquad (a,b,c \in \mathbb{R},\ a \neq 0)$$

Convert to a system:
$$\begin{cases} \dfrac{dy}{dt} = v \\[2mm] \dfrac{dv}{dt} = -\dfrac{c}{a}y - \dfrac{b}{a}v \end{cases}$$

almost the same

Matrix notation:
$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}}_{A} \begin{bmatrix} y \\ v \end{bmatrix} \xrightarrow{\text{solve}} \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{bmatrix}$$
$$= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$

characteristic polynomial $\rightarrow$ $\boxed{a\lambda^2 + b\lambda + c = 0}$

roots of this polynomial are the eigenvalues

CASES:

- Two distinct roots λ_1, λ_2: $\quad y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$
- Repeated root: λ: $\quad y(t) = k_1 e^{\lambda t} + k_2 t e^{\lambda t}$
- Complex conjugate roots: $\lambda = \alpha + i\beta$ $\quad y(t) = k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t)$

EXAMPLES:

1. $y'' - y = 0$ $\qquad$ characteristic polynomial: $\lambda^2 - 1 = 0$

   $y'' = y$

   so $\lambda = \pm 1$

   general solution: $\boxed{y(t) = k_1 e^{t} + k_2 e^{-t}}$

2. $y'' - 2y' + 5y = 0$ $\qquad$ characteristic polynomial: $\lambda^2 - 2\lambda + 5 = 0$

   $\lambda = 1 \pm 2i$

   general solution: $\boxed{y(t) = k_1 e^{t}\cos(2t) + k_2 e^{t}\sin(2t)}$

HARMONIC OSCILLATORS

$$m \frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

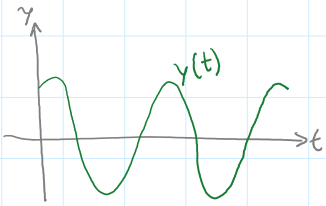
mass $m > 0$ — damping coefficient $b \geq 0$ — spring constant $k > 0$

UNDAMPED: $b = 0$

equation: $my'' + ky = 0$

characteristic polynomial: $m\lambda^2 + k = 0$

$\lambda^2 = \frac{-k}{m}$, so $\underbrace{\lambda = \pm i\sqrt{\frac{k}{m}}}_{\text{pure imaginary}}$



Solution: $y(t) = k_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + k_2 \sin\left(t\sqrt{\frac{k}{m}}\right)$

DAMPED: $b \neq 0$

characteristic polynomial: $m\lambda^2 + b\lambda + k = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

- If $b^2 - 4mk > 0$, then two real roots λ_1 and λ_2, both λ_1 and λ_2 are negative!

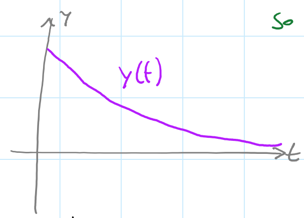
$0 < b^2 - 4mk < b^2$

$\sqrt{b^2-4mk} < b$

so $-b + \sqrt{b^2-4mk} < 0$

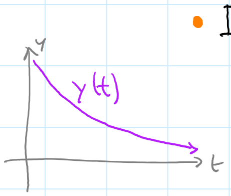
$-b - \sqrt{b^2-4mk} < 0$

Solution: $y(t) = k_1 e^{\overbrace{\lambda_1}^{\text{negative}} t} + k_2 e^{\overbrace{\lambda_2}^{\text{negative}} t}$



OVERDAMPED

- If $b^2 - 4mk = 0$, then repeated root $\lambda = \frac{-b \pm 0}{2m} = \frac{-b}{2m}$



Solution: $y(t) = k_1 e^{\lambda t} + k_2 t e^{\lambda t}$ **CRITICALLY DAMPED**

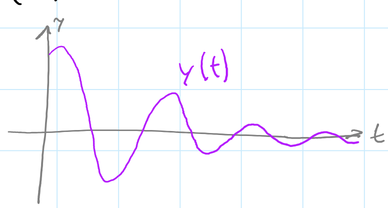
- If $b^2 - 4mk < 0$, then complex conjugate roots $\lambda = \frac{-b \pm i\sqrt{4mk - b^2}}{2m} = \alpha + i\beta$

$\alpha < 0$

Solution: $y(t) = k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t)$

$\alpha = \frac{-b}{2m}$, $\beta = \frac{\sqrt{4mk - b^2}}{2m}$

UNDERDAMPED



WORKSHEET: $y'' + py' + y = 0$

1. **undamped:** $p = 0$, $\lambda = \pm i$, $y(t) = k_1 \cos(t) + k_2 \sin(t)$

2. **underdamped:** $0 < p < 2$, $\lambda = \frac{-p}{2} \pm i \frac{\sqrt{4-p^2}}{2}$,

$$y(t) = e^{\frac{-p}{2}t}\left(k_1 \cos\left(\frac{\sqrt{4-p^2}}{2}t\right) + k_2 \sin\left(\frac{\sqrt{4-p^2}}{2}t\right)\right)$$

3. **critically damped:** $p = 2$, $\lambda = \frac{-p}{2} = -1$, $y(t) = k_1 e^{-t} + k_2 t e^{-t}$

4. **overdamped:** $p > 2$, $\lambda = \frac{-p \pm \sqrt{p^2-4}}{2}$,

$$y(t) = k_1 e^{\frac{-p+\sqrt{p^2-4}}{2}t} + k_2 e^{\frac{-p-\sqrt{p^2-4}}{2}t}$$