EXAMPLE: $y'' + 5y' + 4y = \sin(t)$

- Homogeneous: $y'' + 5y' + 4y = 0$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda+4)(\lambda+1) = 0 \quad \text{so} \quad \lambda = -1 \text{ or } -4$$

sol: $y_h(t) = k_1 e^{-t} + k_2 e^{-4t}$

- Nonhomogeneous: $y'' + 5y' + 4y = \sin(t)$

complexify: $y'' + 5y' + 4y = e^{it}$ sin(t) is the imaginary part of e^{it}

guess: $y_c(t) = Ae^{it}$

plug in: $-A\cancel{e^{it}} + 5Ai\cancel{e^{it}} + 4A\cancel{e^{it}} = \cancel{e^{it}}$

$$-A + 5Ai + 4A = 1$$

$$A(5i+3) = 1$$

$$A = \frac{1}{5i+3} \cdot \frac{5i-3}{5i-3} = \frac{5i-3}{-25-9} = \frac{5i-3}{-34}$$

$$\boxed{A = \frac{3-5i}{34}}$$

so: $y_c(t) = \frac{3-5i}{34} e^{it}$

$$y_c(t) = \frac{3-5i}{34}(\cos t + i \sin t)$$

$$y_c(t) = \frac{1}{34}(3\cos t + 3i \sin t - 5i \cos t + 5 \sin t)$$

$$y_c(t) = \frac{1}{34}(3\cos t + 5\sin t) + i\underbrace{\frac{1}{34}(3\sin t - 5\cos t)}_{\text{imaginary part}}$$

- General solution:

$$\boxed{y(t) = \underbrace{k_1 e^{-t} + k_2 e^{-4t}}_{\text{goes to zero as } t\to\infty} + \underbrace{\frac{1}{34}(3\sin t - 5\cos t)}_{\text{long-term oscillation "steady-state" solution}}}$$

What is its magnitude?

Option 1: trig identity

$$a \cdot \sin(t) + b \cdot \cos(t) = \underbrace{C}_{\text{magnitude}} \cdot \sin(t + \varphi)$$

where $C = \sqrt{a^2+b^2}$ and $\varphi = \arctan\left(\frac{b}{a}\right)$

Here, $a = \frac{3}{34}$, $b = \frac{-5}{34}$, so $C = \sqrt{\left(\frac{3}{34}\right)^2 + \left(\frac{-5}{34}\right)^2} = \boxed{\frac{1}{\sqrt{34}} \approx 0.17}$ magnitude of the steady-state

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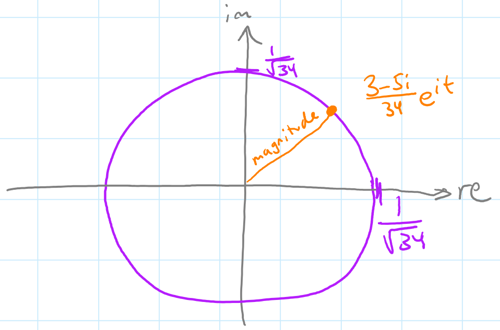
$$y_{in}(t) = \frac{1}{\sqrt{34}} \sin\left(t + \arctan\left(\frac{-5}{3}\right)\right)$$

Option 2: complex magnitude (distance from the origin in the complex plane)

$$|y_c(t)| = \left|\frac{3-5i}{34} e^{it}\right| = \left|\frac{3-5i}{34}\right| \underbrace{|e^{it}|}_{\text{a point on the unit circle in the complex plane}} = \left|\frac{3-5i}{34}\right| = \sqrt{\left(\frac{3}{34}\right)^2 + \left(\frac{-5}{34}\right)} = \frac{1}{\sqrt{34}}$$



Magnitude of the complex solution <u>is</u> the magnitude of the steady-state solution.

WORKSHEET: $y'' + 10y = \cos(\omega t)$ $\quad y(0)=0,\ y'(0)=0$

- Homogeneous: $y'' + 10y = 0$

$\lambda^2 + 10 = 0$ so $\lambda = \pm i\sqrt{10}$

$$\boxed{y_h(t) = k_1 \cos(\sqrt{10}\, t) + k_2 \sin(\sqrt{10}\, t)}$$

- Nonhomogeneous: $y'' + 10y = \cos(\omega t)$

guess: $y_p(t) = A\cos(\omega t)$ so $y_p''(t) = -A\omega^2 \cos(\omega t)$

plug in: $y_p''(t) + 10 y_p(t) = -A\omega^2 \cancel{\cos(\omega t)} + 10A\cancel{\cos(\omega t)} = \cancel{\cos(\omega t)}$

$$A(-\omega^2 + 10) = 1$$

$$A = \frac{1}{10-\omega^2}$$

so $\boxed{y_p(t) = \frac{1}{10-\omega^2}\cos(\omega t)}$ if $\omega^2 \neq 10$

What if $\omega^2 = 10$? e.g. $y'' + 10y' = \cos(\sqrt{10}\, t)$

remember: $y_h(t) = k_1\cos(\sqrt{10}\, t) + k_2 \sin(\sqrt{10}\, t)$

try: $y_p(t) = At\cos(\sqrt{10}\, t)$

to be continued...