Question: What are the equilibrium points of the system?

Nonlinear system!

$$\begin{cases} \dfrac{dx}{dt} = 6x - 2x^2 - y = 0 \Rightarrow y = 6x - 2x^2 \\ \dfrac{dy}{dt} = 5y - xy - y^2 = 0 \Rightarrow y(5-x-y) = 0 \end{cases}$$

$y=0$ or $5-x-y=0$

If $y=0$: $\quad 0 = 6x - 2x^2$

so $0 = 2x(3-x)$ then $x=0$ or $x=3$

If $5-x-y=0$: $\quad y = 5-x$ and $y = 6x-2x^2$

so: $5-x = 6x - 2x^2$

$2x^2 - 7x + 5 = 0$

$(2x-5)(x-1) = 0$

$x = \frac{5}{2}$ or $x = 1$

$y = 5 - \frac{5}{2} = \frac{5}{2}$ $\qquad$ $y = 5-1 = 4$

Equilibrium points

$(0,0)$, $(3,0)$

$\left(\frac{5}{2}, \frac{5}{2}\right)$, $(1,4)$

Question: What are the types of these equilibrium points?

Nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = 6x - 2x^2 - y \\ \dfrac{dy}{dt} = 5y - xy - y^2 \end{cases}$$

these terms go to zero quickly as $(x,y) \to (0,0)$

Linearization

approximated near $(0,0)$ by

$\longrightarrow$

Linear system

$$\begin{cases} \dfrac{dx}{dt} = 6x - y \\ \dfrac{dy}{dt} = 5y \end{cases}$$

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 6 & -1 \\ 0 & 5 \end{bmatrix} \vec{Y}$$

eigenvalues $\lambda_1 = 5$, $\lambda_2 = 6$

equilibrium point is a source

eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(A - 5I) \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \vec{v} = 0$

$(A - 6I) \Rightarrow \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$

What about the equilibrium point (1,4)?

Let $f(x,y) = 6x - 2x^2 - y$

zero, since (1,4) is an equilibrium point

Near $(x,y) = (1,4)$: $f(x,y) \approx \underbrace{f(1,4) + \frac{\partial f}{\partial x}(1,4)\cdot(x-1) + \frac{\partial f}{\partial y}(1,4)\cdot(y-4)}_{\text{tangent plane to } f(x,y) \text{ at } (1,4)}$

[Recall 1D: tangent line: $f(x) \approx f(a) + f'(a)(x-a)$ near $x = a$]

$f(x,y) = 6x - 2x^2 - y$

$g(x,y) = 5y - xy - y^2$

Then:
$$\begin{cases} \frac{dx}{dt} = f(x,y) \approx \frac{\partial f}{\partial x}(1,4)\cdot(x-1) + \frac{\partial f}{\partial y}(1,4)\cdot(y-4) = 2(x-1) + (-1)(y-4) \\ \frac{dy}{dt} = g(x,y) \approx \frac{\partial g}{\partial x}(1,4)\cdot(x-1) + \frac{\partial g}{\partial y}(1,4)\cdot(y-4) = -4(x-1) - 4(y-4) \end{cases}$$

Partial derivatives	Evaluate at (1,4)
$\frac{\partial f}{\partial x} = 6 - 4x$	$\frac{\partial f}{\partial x}(1,4) = 6 - 4(1) = 2$
$\frac{\partial f}{\partial y} = -1$	$\frac{\partial f}{\partial y}(1,4) = -1$
$\frac{\partial g}{\partial x} = -y$	$\frac{\partial g}{\partial x}(1,4) = -4$
$\frac{\partial g}{\partial y} = 5 - x - 2y$	$\frac{\partial g}{\partial y}(1,4) = 5 - 1 - 2(4) = -4$

Linear approximation:
$$\begin{cases} \frac{dx}{dt} = 2x - y + 2 \\ \frac{dy}{dt} = -4x - 4y + 20 \end{cases}$$

The equilibrium point of the nonlinear system at (1,4) has the same type as that of the linear system

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & -1 \\ -4 & -4 \end{bmatrix} \vec{Y}$$

eigenvalues $\lambda = -1 \pm \sqrt{13}$

Saddle

Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

matrix of (first) partial derivatives

evaluate at (1,4)

WORKSHEET: $\begin{cases} \frac{dx}{dt} = x(-x-3y+15) \\ \frac{dy}{dt} = y(-2x-y+10) \end{cases}$

(a) $x(-x-3y+15)=0 \Rightarrow x=0$ or $-x-3y+15=0$

$y(-2x-y+10)=0 \Rightarrow y=0$ or $-2x-y+10=0$

Equilibrium points: $(0,0)$, $(0,10)$, $(15,0)$, $(3,4)$

(b) $f(x,y) = -x^2-3xy+15x$, $\quad g(x,y) = -2xy-y^2+10y$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -2x-3y+15 & -3x \\ -2y & -2x-2y+10 \end{bmatrix}$$

(c) Evaluate J at each equilibrium point:

$J(0,0) = \begin{bmatrix} 15 & 0 \\ 0 & 10 \end{bmatrix}$ $\quad \lambda = 15, 10 \quad$ <u>source</u>

$J(0,10) = \begin{bmatrix} -15 & 0 \\ -20 & -10 \end{bmatrix}$ $\quad \lambda=-15, \lambda=-10 \quad$ <u>sink</u>

$J(15,0) = \begin{bmatrix} -15 & -45 \\ 0 & -20 \end{bmatrix}$ $\quad \lambda = -15, -20 \quad$ <u>sink</u>

$J(3,4) = \begin{bmatrix} -3 & -9 \\ -8 & -4 \end{bmatrix}$ $\quad \lambda = -12, 5 \quad$ <u>saddle</u>