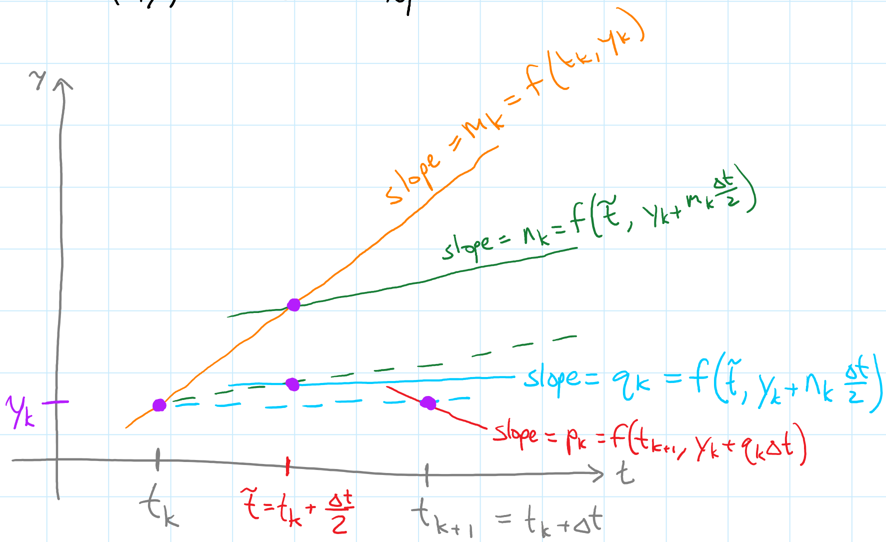
Runge-Kutta Method

$\frac{dy}{dt} = f(t,y)$, $y(0) = y_0$, Approximate $y(a)$.

We will use <u>four</u> values of $f(t,y)$ at each step of the approximation.

1. $m_k = f(t_k, y_k)$ ✱
2. $n_k = f(\tilde{t}, y_k + m_k \frac{\Delta t}{2})$ ✱
3. $q_k = f(\tilde{t}, y_k + n_k \frac{\Delta t}{2})$ ✱
4. $p_k = f(t_{k+1}, y_k + q_k \Delta t)$ ✱



The next approximation y_{k+1} is:

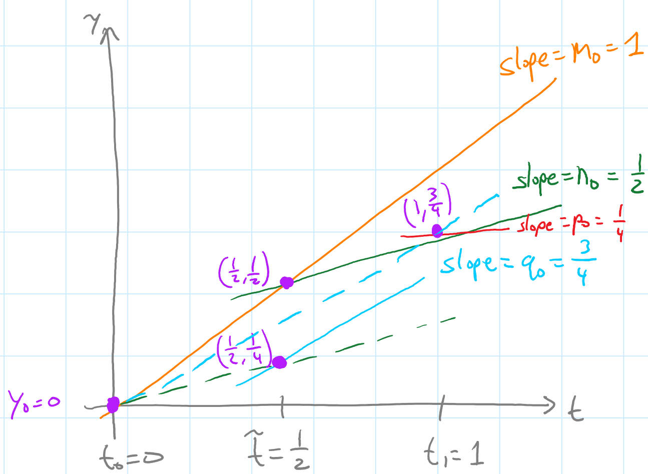
$$y_{k+1} = y_k + \left(\frac{m_k + 2n_k + 2q_k + p_k}{6}\right)\Delta t$$

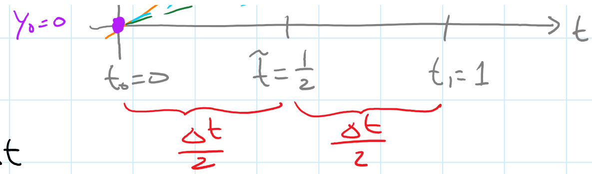
This slope is a weighted average of the four slopes m_k, n_k, q_k, p_k.

Example:

$\frac{dy}{dt} = \underbrace{1 - y}_{f(t,y) = 1-y}$, $y(0) = 0$, approx. $y(1)$ with $\Delta t = 1$.

1. $m_0 = f(0,0) = 1$
2. $n_0 = f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$
3. $q_0 = f(\frac{1}{2}, \frac{1}{4}) = \frac{3}{4}$
4. $p_0 = f(1, \frac{3}{4}) = \frac{1}{4}$





Then: $y_1 = y_0 + \left(\frac{m_0 + 2n_0 + 2q_0 + p_0}{6}\right)\Delta t$

$$y_1 = 0 + \frac{1 + 2\left(\frac{1}{2}\right) + 2\left(\frac{3}{4}\right) + \frac{1}{4}}{6}\cdot 1 = \frac{1 + 1 + \frac{3}{2} + \frac{1}{4}}{6} = \frac{\frac{15}{4}}{6} = \frac{15}{24} = \frac{5}{8} = 0.625$$

So $y(1) \approx 0.625$ ← very close for 1 step of a method

Recall: exact solution is $y(t) = 1 - e^{-t}$, so $y(1) = 1 - e^{-1} = 0.63212$

Runge-Kutta is a fourth-order method:

error is $\underbrace{O(\Delta t^4)}_{\text{not greater that a multiple of } \Delta t^4}$.