REVIEW PROBLEMS

1. $$\frac{dx}{dt} = 10 - x^2 - y^2$$
$$\frac{dy}{dt} = 3x - y$$

(a) Equilibrium points:

$$\frac{dy}{dt} = 0 \Rightarrow y = 3x$$
$$\frac{dx}{dt} = 0 \Rightarrow 10 - x^2 - (3x)^2 = 0$$
$$10 - 10x^2 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$
$$x = 1 \Rightarrow y = 3; \quad x = -1 \Rightarrow y = -3$$

Equilibrium Points: $(1,3)$, $(-1,-3)$

Jacobian: $J(x,y) = \begin{bmatrix} -2x & -2y \\ 3 & -1 \end{bmatrix}$

$J(1,3) = \begin{bmatrix} -2 & -6 \\ 3 & 1 \end{bmatrix}$ has eigenvalues $\lambda = \dfrac{-3 \pm i\sqrt{71}}{2}$

so $(1,3)$ is a spiral sink

$J(-1,-3) = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$ has eigenvalues $\lambda = -4$, $\lambda = 5$

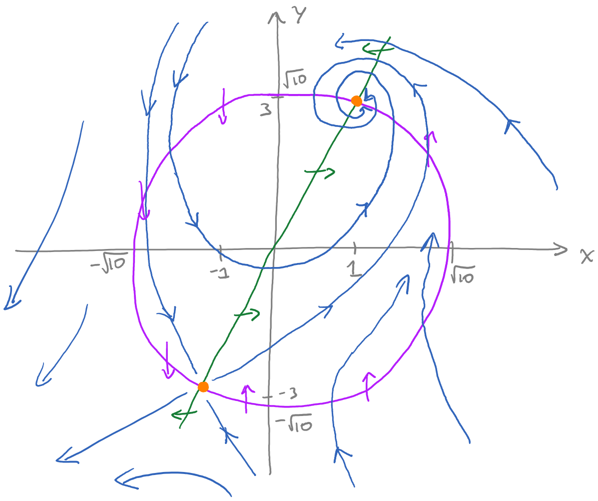
so $(-1,-3)$ is a saddle

(b) Nullclines:

x-nullcline: $x^2 + y^2 = 10$

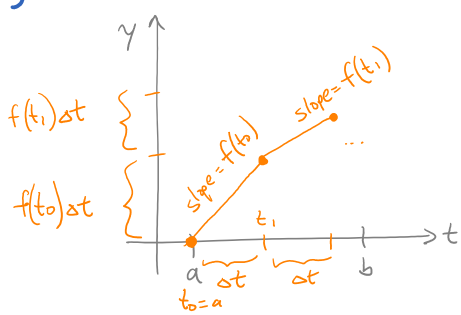
y-nullcline: $y = 3x$

Phase portrait:



2. $$\frac{dy}{dt} = f(t), \quad y(a) = 0$$

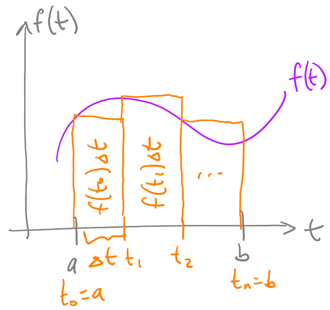
(a) Euler's method:



$$y(b) \approx f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t$$

$$y(b) \approx \sum_{k=0}^{n-1} f(t_k)\Delta t$$

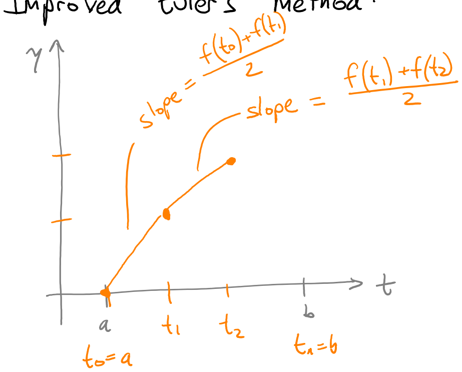
Approximating $\int_a^b f(t)\,dt$ by left Riemann sums



$$\int_a^b f(t)\,dt \approx f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t$$

$$\int_a^b f(t)\,dt \approx \sum_{k=0}^{n-1} f(t_k)\,\Delta t$$

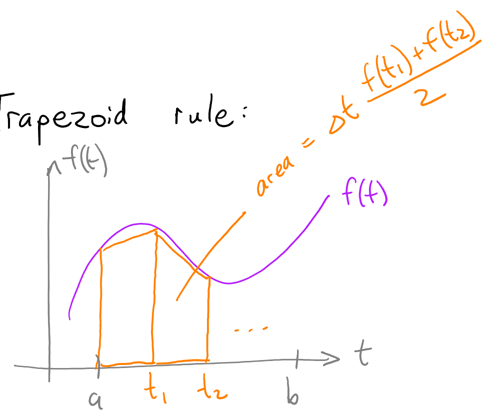
(b) Improved Euler's method:



$$y(b) \approx \Delta t\,\frac{f(t_0)+f(t_1)}{2} + \Delta t\,\frac{f(t_1)+f(t_2)}{2} + \cdots + \Delta t\,\frac{f(t_{n-1})+f(t_n)}{2}$$

$$\approx \boxed{\sum_{k=0}^{n-1} \Delta t\,\frac{f(t_k)+f(t_{k+1})}{2}}$$

Trapezoid rule:



$$\int_a^b f(t)\,dt \approx \Delta t\,\frac{f(t_0)+f(t_1)}{2} + \cdots + \Delta t\,\frac{f(t_{n-1})+f(t_n)}{2}$$

$$\approx \boxed{\sum_{k=0}^{n-1} \Delta t\,\frac{f(t_k)+f(t_{k+1})}{2}}$$