1. Suppose you have an unfair coin that lands on heads only 40% of the time. You flip this coin three times. Let X be the number of heads that appear in three flips of the coin. The probability distribution of X is given by the pmf

$$p(x) = \underbrace{\binom{3}{x}}_{\text{arrange } x \text{ heads in 3 flips this many ways}} \underbrace{(0.4)^x}_{x \text{ heads}} \underbrace{(0.6)^{3-x}}_{3-x \text{ tails}} \quad \text{for} \quad x \in \{0, 1, 2, 3\}.$$

Probabilities: $p(0) = 0.6^3 = 0.216$ $p(2) = 3\cdot(0.4)^2(0.6) = 0.288$

$p(1) = 3\cdot(0.4)(0.6)^2 = 0.432$ $p(3) = (0.4)^3 = 0.064$

(a) What is the expected value $E(X)$?

"AVERAGE VALUE"

DEF: $E(X) = \mu_X = \underbrace{\sum_x x \cdot p(x)}_{\text{sum of values multiplied by their probabilities}}$

$$E(X) = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2$$

(b) What is $E(X^2)$?

EXPECTED VALUE OF $f(X)$: $E(f(X)) = \sum_x \underbrace{f(x)}_{\text{value of the function of } x} \cdot \underbrace{p(x)}_{\text{probability that } X = x}$

$$E(X^2) = \sum_{x=0}^{3} x^2 \cdot p(x) = 0^2(0.216) + 1^2(0.432) + 2^2(0.288) + 3^2(0.064) = 2.16$$

(c) What is Var(X)? *Hint: use the shortcut formula!*

VARIANCE: $\text{Var}(X) = \sigma_X^2 = \sum_x \left((x-\mu_X)^2 p(x)\right) = E\left((X-\mu_X)^2\right) = \boxed{E(X^2) - (E(X))^2}$ SHORTCUT:

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 2.16 - (1.2)^2 = 0.72$$

STANDARD DEVIATION: $\sigma_X = \sqrt{\text{Var}(X)}$

(d) Suppose the coin flips are part of a game in which you win $5X + 2$ dollars. Let $Y = 5X + 2$. What is the pmf of Y?

$Y = f(X)$

values of Y	2	7	12	17
probabilities	0.216	0.432	0.288	0.064

$\leftarrow p_Y(y)$

$P(Y=2) = P(X=0)$

(e) Use the pmf of Y to find $E(Y)$, your expected winnings in this game.

$$E(Y) = 2(0.216) + 7(0.432) + 12(0.288) + 17(0.064) = 8$$

(f) Use the pmf of Y to find $E(Y^2)$, and then find Var(Y).

$E(Y^2) = 2^2(0.216) + 7^2(0.432) + 12^2(0.288) + 17^2(0.064) = 82$

$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = 82 - 8^2 = 82 - 64 = 18$

(g) How is $E(Y)$ related to $E(X)$? How is $\mathrm{Var}(Y)$ related to $\mathrm{Var}(X)$?

$Y = 5X + 2$

$E(Y) = 5E(X) + 2$

$8 = 5(1.2) + 2$

$\boxed{\mathrm{Var}(Y) \neq 5\,\mathrm{Var}(X) + 2}$!

$\mathrm{Var}(Y) = 5^2\,\mathrm{Var}(X)$

EXPECTED VALUE IS LINEAR!

$E(aX+b) = aE(X) + b$

Variance is ~~NOT~~ linear!

Instead: $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

why?

$\mathrm{Var}(aX+b) = E\left((aX+b)^2\right) - \left(E(aX+b)\right)^2$

$= E\left(a^2X^2 + 2abX + b^2\right) - \left(aE(X) + b\right)^2$

$= a^2E(X^2) + \cancel{2abE(X)} + \cancel{b^2} - \left(a^2(E(X))^2 + \cancel{2abE(X)} + \cancel{b^2}\right)$

$= a^2\left(E(X^2) - (E(X))^2\right) = a^2\,\mathrm{Var}(X)$

Chebyshev's Inequality: Let X be a discrete random variable with mean μ and standard deviation σ. For any $k \geq 1$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

In words: The probability that (X is at least k standard deviations away from its mean) is at most $\frac{1}{k^2}$.

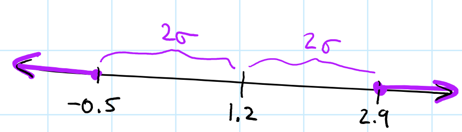
Prob. that X is or is at most $\frac{1}{k^2}$.

2. Verify that Chebyshev's Inequality holds for the random variable X from Problem 1, using the value $k = 2$. That is, check that $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

plug in: $P\left(|X - 1.2| \geq 2\sqrt{0.72}\right) \overset{?}{\leq} \frac{1}{2^2}$

$P\left(|X - 1.2| \geq 1.7\right) \leq \frac{1}{4}$

$P\left(X \leq -0.5 \text{ or } X \geq 2.9\right) \leq \frac{1}{4}$



−0.5 1.2 2.9

$P(X \le -0.5 \text{ or } X \ge 2.9) \le \frac{1}{4}$

since $X \in \{0,1,2,3\}$

$P(X \le -0.5) + P(X \ge 2.9) \le \frac{1}{4}$

$0 + P(X = 3) \le \frac{1}{4}$

$0.064 \le \frac{1}{4}$ TRUE!

3. The number of equipment breakdowns in a manufacturing plant averages 4 per week, with standard deviation 0.7 per week.

(a) Find an interval that includes at least 90% of the weekly figures for the number of breakdowns.

Think about this for next time!

(b) A plant supervisor promises that the number of breakdowns will rarely exceed 7 in a one-week period. Is the supervisor justified in making this claim? Why?