R functions for geometric and neg. binomial rv:

$P(X=x) = \text{dgeom}(x-1, p)$ ← pmf for a geometric rv with prob. p

$P(X \le x) = \text{pgeom}(\underbrace{x-1}_{}, p)$ ← cdf " " " "

└ number of failures before the first success

$\text{dnbinom}(x-r, r, p)$ ← pmf of $X \sim NB(r,p)$

$\text{pnbinom}(\underbrace{x-r}_{}, r, p)$ ← cdf " " "

number of failures before the r^{th} success

A **GENERATING FUNCTION** is a power series whose coefficients encode a sequence of numbers.

example: $1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots = \dfrac{x}{1-x-x^2}$

ordinary generating function of the Fibonacci sequence

Generating functions are useful for working with sequences:

- Finding the terms of a sequence
- Finding recurrence relationships
- Studying asymptotics
- Solving combinatorial problems

To learn more, take Math 364 Combinatorics in the spring!

An **EXPONENTIAL GENERATING FUNCTION** encodes a sequence $a_0, a_1, a_2, \ldots$ by the power series

$$\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$

$\boxed{\textbf{RECALL: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$

The **MOMENT GENERATING FUNCTION (mgf)** of a random variable X is:

$$M_X(t) \overset{def}{=} E(e^{tX}) = \sum_{x} \underbrace{e^{tx}}_{\text{values}} \underbrace{P(X=x)}_{\text{probabilities}}$$

↑ sum is over all possible values of X

This is an exponential generating function for the sequence of moments:

$$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{3!} + \cdots + E(X^n)\frac{t^n}{n!} + \cdots$$

moments of X

EXAMPLE: Let X be a Bernoulli rv with $p(1) = p$. Then:

$$M_X(t) = E(e^{tX}) = e^{t\cdot 0}(1-p) + e^{t\cdot 1}(p) = 1 - p + pe^t$$

$e^t = \sum^{\infty} t^k$

$$M_X(t) = E(e^{tX}) = e^{t\cdot 0}(1-p) + e^{t\cdot 1}(p) = 1-p+pe^t$$

$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$

As a power series:

$$M_X(t) = 1-p+p\sum_{k=0}^{\infty}\frac{t^k}{k!} = 1-p+p\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\cdots\right)$$

$$= \underbrace{1-p+p}_{E(X^0)=E(1)=1.} + \boxed{p}t + \boxed{p}\frac{t^2}{2} + \boxed{p}\frac{t^3}{6} + p\frac{t^4}{24} + \cdots$$

$E(X) = 0(1-p)+1(p) = p$

$E(X^2) = 0^2(1-p)+1^2(p) = p$

$E(X^3)$

WHY MGFs?

1. If we find $M_X(t)$, then we can easily obtain the moments of X, $E(X^n)$ for any n.

   To obtain $E(X^n)$, differentiate $M_X(t)$ n times, and evaluate at $t=0$.

2. If the mgfs of X and Y are equal, then X and Y have the same distribution.

WORKSHEET:

1. Let $X \sim \text{Bin}(n,p)$. We will find $M_X(t)$.

(a) First, what is $E(e^{tX})$? Write this as a sum.

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{n} e^{tk} \underbrace{\binom{n}{k}p^k(1-p)^{n-k}}_{\text{binomial pmf}}$$

(b) Now use the binomial theorem $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$ to simplify your sum from part (a). You will then have a concise expression for $M_X(t)$.

$$M_X(t) = \sum_{k=0}^{\infty}\binom{n}{k}(pe^t)^k(1-p)^{n-k} = (pe^t+1-p)^n$$

$a = pe^t \qquad b = 1-p$

(c) Use your result from part (b), together with the fact that $E(X^r) = M_X^{(r)}(0)$, to find $E(X)$ and $\text{Var}(X)$. Do these agree with the formulas we learned previously?

differentiate r times

$$M_X(t) = (pe^t+1-p)^n$$

$$M_X'(t) = n(pe^t+1-p)^{n-1}(pe^t), \quad \text{so} \quad M_X'(0) = n(pe^0+1-p)^{n-1}pe^0 = n(1)^{n-1}p = np = E(X)$$

$$M_X''(t) = n(n-1)(pe^t+1-p)^{n-2}(pe^t)^2 + n(pe^t+1-p)^{n-1}(pe^t)$$

$$\text{so} \quad M_X''(0) = n(n-1)(p+1-p)^{n-2}(p)^2 + n(p+1-p)^{n-1}(p) = n(n-1)p^2+np = E(X^2)$$

$$\text{Var}(X) = E(X^2)-E(X)^2 = n(n-1)p^2+np-(np)^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np-np^2$$

$$= np(1-p) \qquad \text{yes!}$$

2. Suppose random variable X has probability mass function $P(X=x) = \left(\frac{27}{40}\right)\left(\frac{1}{3}\right)^x$, for integers $0 \le x \le 3$.

(a) Verify that this is a valid probability mass function. to be continued...

(b) Compute the expected value of X from the pmf.

(c) Find the moment generating function $M_X(t)$. If you think for a moment (ha!), it is possible to write $M_X(t)$ without using any summation signs or addition symbols.

(d) Compute $M_X'(0)$. Does your answer agree with your answer for part (b)?