RECALL: moment-generating function (mgf) of a random variable $X$ is defined $M_X(t) = E(e^{tX}) = \sum_x e^{tx} P(X=x)$

Equivalent to: $M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{3!} + \dots + E(X^n)\frac{t^n}{n!} + \dots$

→ To recover $E(X^r)$, differentiate $M_X(t)$ $r$ times and evaluate at $t=0$.

→ If rvs $X$ and $Y$ have the same mgfs, then they have the same distribution.

2. Suppose random variable $X$ has probability mass function $P(X = x) = \frac{27}{40}\left(\frac{1}{3}\right)^x$, for integers $0 \le x \le 3$.

(a) Verify that this is a valid probability mass function.

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{a(1-r^n)}{1-r}$$

- probabilities are not negative
- sum is: $\frac{27}{40}\underbrace{\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right)}_{\substack{\text{finite geometric sum} \\ r=\frac{1}{3}}} = \frac{27}{40}\left(\frac{1\left(1-\left(\frac{1}{3}\right)^4\right)}{1-\frac{1}{3}}\right) = \frac{27}{40}\left(\frac{1-\frac{1}{81}}{1-\frac{1}{3}}\right) = \frac{27}{40}\left(\frac{\frac{80}{81}}{\frac{2}{3}}\right) = \frac{\cancel{27}}{\cancel{40}} \cdot \frac{\cancel{80}}{\cancel{81}} \cdot \frac{\cancel{3}}{\cancel{2}} = 1$

(b) Compute the expected value of $X$ from the pmf.

$$E(X) = 0 \cdot \frac{27}{40} + 1 \cdot \frac{9}{40} + 2 \cdot \frac{3}{40} + 3 \cdot \frac{1}{40} = \frac{9+6+3}{40} = \frac{18}{40} = \frac{9}{20}$$

same!

(c) Find the moment generating function $M_X(t)$. If you think for a moment (ha!), it is possible to write $M_X(t)$ without using any summation signs or addition symbols.

$$M_X(t) = E(e^{tX}) = \underbrace{e^{t\cdot 0}\frac{27}{40} + e^{t\cdot 1}\cdot\frac{9}{40} + e^{t\cdot 2}\frac{3}{40} + e^{t\cdot 3}\frac{1}{40}}_{\text{finite geometric sum: } a=\frac{27}{40},\ r=\frac{e^t}{3},\ n=4}$$

$$= \frac{27}{40}\left(\frac{1-\left(\frac{e^t}{3}\right)^4}{1-\frac{e^t}{3}}\right)\left(\frac{81}{81}\right) = \frac{\cancel{27}}{40}\left(\frac{81-e^{4t}}{\underset{3}{\cancel{81}}-\cancel{27}e^t}\right) = \frac{81-e^{4t}}{40(3-e^t)}$$

(d) Compute $M_X'(0)$. Does your answer agree with your answer for part (b)?

$$M_X(t) = \frac{81-e^{4t}}{40(3-e^t)}$$

$$M_X'(t) = \frac{40(3-e^t)(-4e^{4t}) - (81-e^{4t})(40)(-e^t)}{40^2\,(3-e^t)^2}$$

$$M_X'(0) = \frac{\cancel{40}(3-1)(-4) - (81-1)\cancel{(40)}(-1)}{40^{\cancel{2}}\,(3-1)^2} = \frac{-8+80}{40(4)} = \frac{72}{160} = \frac{9}{20}$$

1. Let $X$ be a discrete random variable with mgf $M_X(t)$, and let $Y = aX + b$.

(a) Show that the mgf of $Y$ is $e^{tb}M_X(at)$.

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(aX+b)}\right) = E(e^{tb}e^{atX}) = e^{tb}E(e^{(at)X}) = e^{tb}M_X(at)$$

Linearity of Expected Value

$$\boxed{\text{If } Y=aX+b, \text{ then } M_Y(t) = e^{bt}M_X(at).}$$

(b) Use the mgf of $Y$ to show that $E(Y) = aE(X) + b$ and $\text{Var}(Y) = a^2\text{Var}(X)$.

differentiate: $M'_Y(t) = be^{bt}M_X(at) + e^{bt}M'_X(at)\,a$

$M'_Y(t) = e^{bt}\left(b\,M_X(at) + a\,M'_X(at)\right)$

So: $M'_Y(0) = e^0\left(b\,\underbrace{M_X(0)}_{1} + a\,M'_X(0)\right)$

Thus: $E(Y) = b + a\,E(X)$

similarly: $M''_Y(t) = e^{bt}\left(b^2M_X(at) + 2ab\,M'_X(at) + a^2M''_X(at)\right)$

so: $E(Y^2) = M''_Y(0) = b^2M_X(0) + 2ab\,M'_X(0) + a^2M''_X(0) = b^2 + 2ab\,E(X) + a^2E(X^2)$

Thus: $\text{Var}(Y) = E(Y^2) - E(Y)^2 = b^2 + 2ab\,E(X) + a^2E(X^2) - (b + aE(X))^2$

$= a^2E(X^2) - a^2E(X)^2 = a^2\,\text{Var}(X)$

2. If $X \sim \text{Poisson}(\mu)$, show that $M_X(t) = e^{\mu(e^t-1)}$.

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$

$P(X=x) = e^{-\mu}\frac{\mu^x}{x!}$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}e^{-\mu}\frac{\mu^x}{x!} = e^{-\mu}\sum_{x=0}^{\infty}e^{tx}\frac{\mu^x}{x!} = e^{-\mu}\boxed{\sum_{x=0}^{\infty}\frac{(e^t\mu)^x}{x!}}$$

$$= e^{-\mu}e^{\mu e^t} = e^{\mu e^t - \mu} = \boxed{e^{\mu(e^t-1)}}$$

3. Let $X$ represent the number of insurance policies sold by an agent in a day. The moment generating function of $X$ is $M_X(t) = 0.45e^t + 0.35e^{2t} + 0.15e^{3t} + 0.05e^{4t}$, for $-\infty < t < \infty$. Calculate the standard deviation of $X$.

We will do this on Monday.

↙ This problem is challenging ... and fun!

4. The monthly amount of time $X$ (in hours) during which a manufacturing plant is inoperative due to equipment failures or power outages follows approximately a distribution with the following moment generating function:

$$M_X(t) = \left(\frac{1}{1 - 7.5t}\right)^2$$

The amount of loss in profit due to the plant being inoperative is given by $Y = 12X + 1.25X^2$. Determine the variance of the loss in profit.