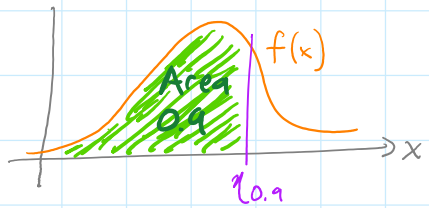
PERCENTILE: For $p \in [0,1]$, the $(100p)^{th}$ percentile is the value η_p such that $P(X \le \eta_p) = p$. Equivalently, $p = \int_{-\infty}^{\eta_p} f(x)\,dx$, or $F(\eta_p) = p$.

Example: $p = 0.9$ means the 90th percentile

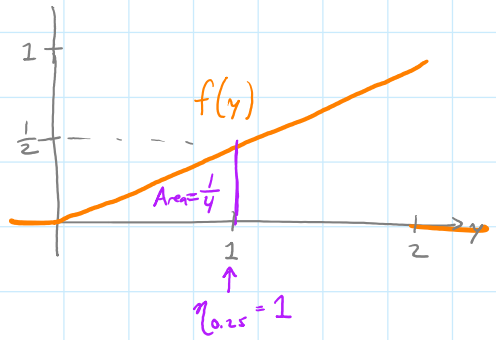


5. Let Y be a random variable with pdf given by $f(y) = \begin{cases} \frac{y}{2} & \text{if } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$

(a) Find a value $\eta_{0.25}$ such that $P(Y \le \eta_{0.25}) = 0.25$.

Note: $p = 0.25$

$$\frac{1}{4} = \int_{-\infty}^{\eta_{0.25}} f(y)\,dy$$



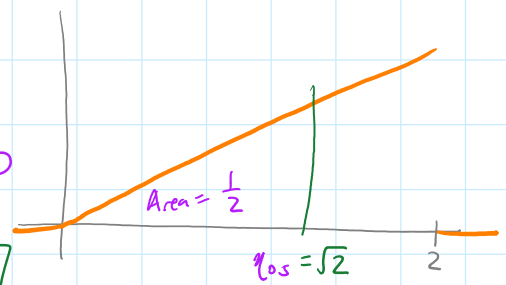
(b) What is the median of Y?

↳ 50th percentile: $p = 0.5$

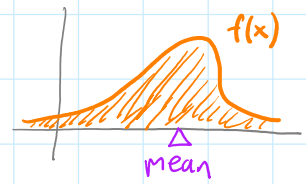
$$\frac{1}{2} = \int_{-\infty}^{\eta_{0.5}} f(y)\,dy = \int_0^{\eta_{0.5}} \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^{\eta_{0.5}} = \frac{\eta_{0.5}^2}{4} - 0$$

$$\frac{1}{2} = \frac{\eta_{0.5}^2}{4} \Rightarrow 2 = \eta_{0.5}^2 \Rightarrow \boxed{\sqrt{2} = \eta_{0.5}}$$

MEDIAN



	Discrete rv X with pmf $p(x)$	Continuous rv X with pdf $f(x)$
mean	$\mu = E(X) = \sum_x x\,p(x)$	$\mu = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$
"Law of the Unconscious Statistician"	$E(h(X)) = \sum_x h(x)\,p(x)$	$E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x)\,dx$
Variance	$Var(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x)$	$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$
moment generating function	$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$	$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

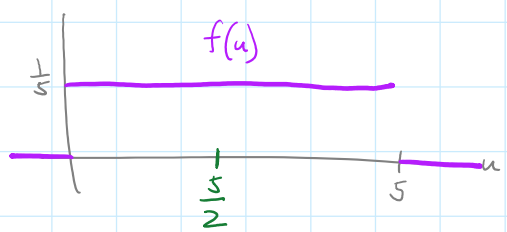


The mean gives the "balance point" of the distribution

FORMULAS THAT APPLY TO BOTH DISCRETE & CONTINUOUS:

$$Var(X) = E(X^2) - E(X)^2, \quad E(aX+b) = a\,E(X) + b, \quad Var(aX+b) = a^2\,Var(X)$$

1. Let $U \sim$ Unif[0,5]. — pdf is $f(u) = \frac{1}{5}$ on $[0,5]$



(a) What are the mean and variance of U?

Mean: $E(U) = \int_0^5 u\,f(u)\,du = \int_0^5 \frac{u}{5}\,du = \frac{u^2}{10}\Big|_0^5 = \frac{25}{10} = \frac{5}{2}$

$E(U^2) = \int_0^5 u^2 f(u)\,du = \int_0^5 \frac{u^2}{5}\,du = \frac{u^3}{15}\Big|_0^5 = \frac{125}{15} = \frac{25}{3}$

Variance: $Var(U) = E(U^2) - E(U)^2 = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{3} - \frac{25}{4} = \frac{100-75}{12} = \frac{25}{12} = \frac{(B-A)^2}{12}$

Variance of a Unif. rv. on $[A,B]$

(b) Let $V = 3U + 2$. What are the mean and variance of V?

mean: $E(V) = E(3U+2) = 3E(U) + 2 = 3\frac{5}{2} + 2 = \frac{15}{2} + 2 = \frac{19}{2}$

var: $Var(V) = Var(3U+2) = 3^2 Var(U) = 9 \cdot \frac{25}{12} = \frac{75}{4}$

(c) What kind of distribution does V have? How do you know?

tricky

$V \sim$ Unif$[2, 17]$

We'll return to this...

2. Let Y be a random variable with pdf given by $f(y) = \begin{cases} \frac{y}{2} & \text{if } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$

(a) What is the expected value of Y?

$$E(Y) = \int_0^2 y\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Big|_0^2 = \frac{2^3}{6} - \frac{0^3}{6} = \frac{8}{6} = \frac{4}{3}$$

(b) What is $E(Y^2)$?

$$E(Y^2) = \int_0^2 y^2\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^3}{2}\,dy = \frac{y^4}{8}\Big|_0^2 = \frac{2^4}{8} - 0 = 2$$

(c) What is $\mathrm{Var}(Y)$?

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 4 - \frac{16}{9} = \frac{2}{9}$$

$\int u\,dv = uv - \int v\,du$

(d) What is the moment generating function $M_Y(t)$?

$$M_Y(t) = E(e^{tY}) = \int_0^2 e^{ty}\frac{y}{2}\,dy = \frac{1}{2}\int_0^2 \underbrace{y}_{u}\,\underbrace{e^{ty}\,dy}_{dv} = \frac{1}{2}\left(\left[y\cdot\frac{1}{t}e^{ty}\right]_0^2 - \int_0^2 \frac{1}{t}e^{ty}\,dy\right)$$

If $t \neq 0$: $u = y$, $du = dy$; $v = \frac{1}{t}e^{ty}$, $dv = e^{ty}\,dy$

$$= \frac{1}{2}\left[\frac{y}{t}e^{ty} - \frac{1}{t^2}e^{ty}\right]_0^2 = \frac{1}{2}\left[\frac{2}{t}e^{2t} - \frac{1}{t^2}e^{2t} - \left(0 - \frac{1}{t^2}\right)\right]$$

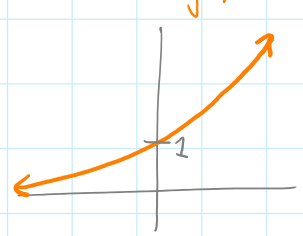
$$= \frac{1}{2}\left[e^{2t}\left(\frac{2}{t} - \frac{1}{t^2}\right) + \frac{1}{t^2}\right] = e^{2t}\frac{2t-1}{2t^2} + \frac{1}{2t^2}$$

If $t = 0$: $M_Y(0) = \int_0^2 e^0\frac{y}{2}\,dy = \int_0^2 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^2 = 1$

Thus:

$$M_Y(t) = \begin{cases} 1 & \text{if } t = 0 \\ e^{2t}\left(\frac{2t-1}{2t^2}\right) + \frac{1}{2t^2} & \text{if } t \neq 0 \end{cases}$$

graph of $M_Y(t)$



Note: continuous at $t=0$

3. Let $X \sim \mathrm{Unif}[A, B]$. Show that the mgf of X is $M_X(t) = \begin{cases} \frac{e^{Bt} - e^{At}}{(B-A)t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$.

Then use properties of mgfs to verify your answer for 1(c).