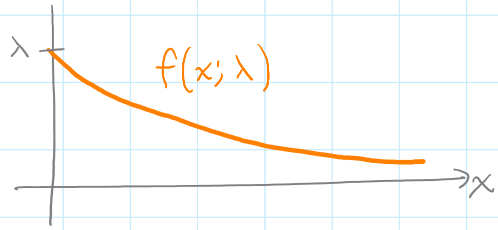
EXPONENTIAL DISTRIBUTION

The times between events in a Poisson process are exponentially distributed.

pdf: $X \sim Exp(\lambda)$ has pdf $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$



1. Let $X \sim \mathrm{Exp}(\lambda)$. Show that $E(X) = \frac{1}{\lambda}$.

$$E(X) = \int_0^\infty \underset{\text{value}}{x} \, \underbrace{\lambda e^{-\lambda x}}_{\text{density}} dx = \lambda \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\lambda x} dx}_{dv} = \lambda \left[\overset{uv}{\frac{-1}{\lambda} x e^{-\lambda x} \Big|_0^\infty} - \overset{\int v\, du}{\int_0^\infty \frac{-1}{\lambda} e^{-\lambda x} dx} \right]$$

$u = x$ $\quad v = \frac{-1}{\lambda} e^{-\lambda x}$

$du = dx$ $\quad dv = e^{-\lambda x} dx$

$$= \lambda \left[\frac{-1}{\lambda} x e^{-\lambda x} \Big|_0^\infty - \frac{1}{\lambda^2} e^{-\lambda x} \Big|_0^\infty \right] = \lambda \left[0 - \left(0 - \frac{1}{\lambda^2} \right) \right] = \boxed{\frac{1}{\lambda}}$$

same!

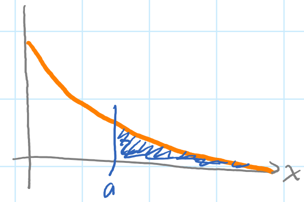
Also: $E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \ldots$ integrate by parts twice $\ldots = \frac{2}{\lambda^2}$

Then: $Var(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

So the standard deviation is $\sigma_X = \sqrt{Var(X)} = \boxed{\frac{1}{\lambda}}$

2. Let $X \sim \mathrm{Exp}(\lambda)$ and $0 < a < b$.

(a) What is $P(X \geq a)$?



$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_a^\infty = 0 + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

exponential tail probability

(b) Show that $P(X > b \mid X > a) = P(X > b - a)$.

$$P(X > b \mid X > a) = \frac{P(X > b \text{ and } X > a)}{P(X > a)} = \frac{P(X > b)}{P(X > a)} = \frac{e^{-\lambda b}}{e^{-\lambda a}} = e^{-\lambda(b-a)}$$

$$= P(X > b - a)$$

(c) What other distribution satisfies the equality in (b)?

geometric! Memoryless Property

(d) Convince yourself that the property in (b) is special, in the sense that it doesn't hold for most random variables. For example, if $U \sim$ Unif[0,10], show that $P(U > 4 \mid U > 3) \neq P(U > 1)$.

$$P(U>4 \mid U>3) = \frac{P(U>4)}{P(U>3)} = \frac{\frac{6}{10}}{\frac{7}{10}} = \frac{6}{7}$$

$$P(U > 4-3) = P(U>1) = \frac{9}{10}$$

NOT THE SAME

3. Suppose that emails arrive in your inbox according to a Poisson process with rate 2 emails per hour. Then the time between successive emails is an exponential random variable with mean 30 minutes.

(a) What is the probability that you don't receive any emails in the next hour?

EXPONENTIAL: $X \sim Exp(2)$ — $\lambda = 2$, so $E(X) = \frac{1}{\lambda} = \frac{1}{2}$ hour

time to the next email

so $P(X>1) = \int_1^\infty 2e^{-2x}\,dx = -e^{-2x}\Big|_1^\infty = \boxed{e^{-2}}$

POISSON: $Y \sim Poisson(2)$ — $E(Y) = 2$

number of emails in the next hour

so $P(Y=0) = e^{-2}\frac{2^0}{0!} = \boxed{e^{-2}}$

(b) What is the standard deviation of the time until the next email?

$$\sigma_X = E(X) = \frac{1}{2} \text{ hour}$$

MOMENT-GENERATING FUNCTION of $Exp(\lambda)$

Assume $t<\lambda$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda}e^{(t-\lambda)x}\Big|_0^\infty$$

$$= \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda-t}$$

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$$

If $X \sim Exp(1)$, then: $E(X) = 1$, $E(X^2) = 2$, $E(X^3) = 6$,

$E(X^4) = 24$, $E(X^5) = 120, \ldots$

$E(X^n) = n!$