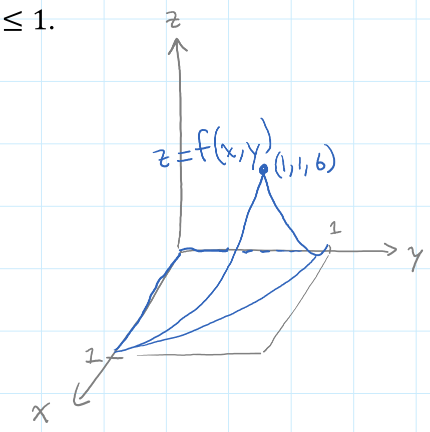
1. Let X and Y have joint pdf $f(x,y) = 6xy^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

Check that this is a pdf: $f(x,y) \ge 0$

$$\int_0^1\int_0^1 6xy^2 \, dx\, dy = 1$$



(a) What is $E(X+Y)$?

The expected value of $h(X,Y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underset{\text{values}}{h(x,y)}\ \underset{\text{density}}{f(x,y)} \, dx\, dy$$

Then:

$$E(X+Y) = \int_0^1\int_0^1 (x+y)\, 6xy^2 \, dx\, dy = \frac{17}{12}$$

(b) What is $E(XY)$?

$$E(XY) = \int_0^1\int_0^1 (xy)\, 6xy^2 \, dx\, dy = \frac{1}{2}$$

(c) What are $E(X)$ and $E(Y)$?

Marginal density of X: $f_X(x) = \int_0^1 6xy^2\, dy = 2xy^3\Big|_{y=0}^{y=1} = 2x$, for $0 \le x \le 1$

So $E(X) = \int_0^1 x \cdot f_X(x)\, dx = \int_0^1 x \cdot 2x\, dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

or: $E(X) = \int_0^1\int_0^1 (x)\, 6xy^2\, dy\, dx = \int_0^1 x \boxed{\int_0^1 6xy^2\, dy}\, dx = \frac{2}{3}$

$$E(Y) = \int_0^1\int_0^1 y \cdot 6xy^2\, dx\, dy = \frac{3}{4}$$

$f_Y(y) = 3y^2$

(d) How do $E(X)$ and $E(Y)$ relate to $E(X+Y)$ and $E(XY)$?

Linearity of expected value

$$\boxed{E(X) + E(Y) = E(X+Y)}$$

$$\frac{2}{3} + \frac{3}{4} = \frac{17}{12}$$

$$E(X)\,E(Y) = E(XY)$$

$$\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$$

$\boxed{\begin{array}{l}\text{If } X \text{ and } Y \text{ are independent,} \\ \text{then } E(XY) = E(X)E(Y). \\ \text{The converse is not true!}\end{array}}$!

(e) What is Cov(X, Y)?

$$\text{Cov}(X,Y) = E\left((X-\mu_X)(Y-\mu_Y)\right)$$

Shortcut:

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

Here, $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \frac{2}{3}\cdot\frac{3}{4} = 0$

X and Y are uncorrelated

CORRELATION COEFFICIENT:

$$\text{Corr}(X,Y) = \rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

standard deviations

Correlation is independent of units of measurement.

$$-1 \le \rho_{X,Y} \le 1$$

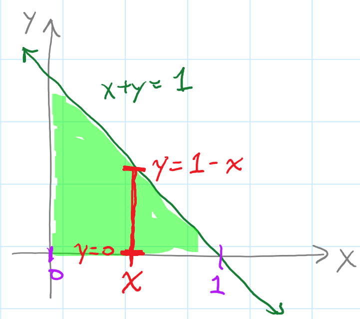
- Independent variables are uncorrelated.
- Uncorrelated variables might be dependent

see §4.2 in the text

2. Now let X and Y have joint pdf $f(x, y) = 3x + 3y$ for $0 \le x$, $0 \le y$, and $x + y \le 1$. 1

(a) What is $E(X + Y)$?

$$E(X+Y) = \int_0^1\int_0^{1-x} (x+y)(3x+3y)\,dy\,dx = \frac{3}{4}$$



(b) What is $E(XY)$?

$$E(XY) = \int_0^1\int_0^{1-x} xy(3x+3y)\,dy\,dx = \frac{1}{10}$$

(c) What are $E(X)$ and $E(Y)$?

$$f_X(x) = \int_0^{1-x}(3x+3y)\,dy = \frac{3}{2}(1-x^2), \quad 0 \le x \le 1, \quad \text{so} \quad E(X) = \int_0^1 x\cdot\frac{3}{2}(1-x^2)\,dx = \frac{3}{8}$$

$$f_Y(y) = \int_0^{1-y}(3x+3y)\,dx = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1, \quad \text{so} \quad E(Y) = \int_0^1 y\cdot\frac{3}{2}(1-y^2)\,dy = \frac{3}{8}$$

(d) How do $E(X)$ and $E(Y)$ relate to $E(X+Y)$ and $E(XY)$?

$E(X+Y) = E(X) + E(Y)$, but $E(XY) \neq E(X)E(Y)$

linearity of expectation

here, X and Y are <u>not</u> independent

(e) What is $\text{Cov}(X, Y)$?

shortcut formula: $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{10} - \frac{3}{8}\cdot\frac{3}{8} = \frac{1}{10} - \frac{9}{64} = -\frac{13}{320}$