STURM – LIOUVILLE THEORY

Generalization of the eigenvalue problems that we have studied.

STURM-LIOUVILLE EQUATION: $\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a<x<b$

Here, ϕ is the unknown function and λ is the unknown eigenvalue, p, q, σ are known functions, $p(x)>0$ and $\sigma(x)>0$.

REGULAR: $\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$ $\quad\quad\beta_i$ is any real constant

$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0$

For any regular Sturm-Liouville equation, 6 theorems hold — p. 157 in the text

EXAMPLE: $\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0$

This is a regular S-L equation: $p(x)=1,\ q(x)=0,\ \sigma(x)=1$

boundary conditions: $a=0,\ b=L,\ \beta_1=1,\ \beta_2=0,\ \beta_3=1,\ \beta_4=0$

THEOREMS

1. All eigenvalues are real.

   We know: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n\in\{1,2,3,\dots\}$

   real numbers

   We didn't look for non-real eigenvalues, but there aren't any.

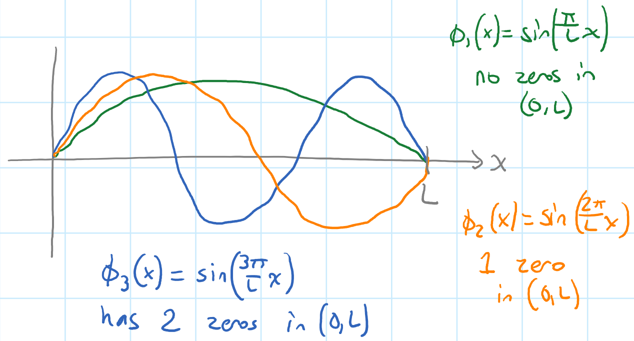
2. There is an infinite sequence of eigenvalues, with a smallest, but no largest eigenvalue.

   Yes: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n\in\{1,2,3,\dots\}$

3. There is a unique eigenfunction ϕ_n for each eigenvalue, and ϕ_n has $n-1$ zeros in $a<x<b$.

   $\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ is the unique (up to multiplicative constant) eigenfunction for λ_n

$\sin\left(\frac{n\pi}{L}x\right)$ has $n-1$ zeros in $0<x<L$



4. Any piecewise smooth $f(x)$ has a Fourier series in terms of the eigenfunctions.

Yes: any piecewise smooth $f(x)$ has a Fourier sine series.

5. Eigenfunctions are orthogonal:

We know: $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0$ if $n\neq m$.

6. Rayleigh Quotient:
$$\lambda = \frac{\left[-p\phi\frac{d\phi}{dx}\right]_a^b + \int_a^b\left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\, dx}$$

Here, $p=1$, $\sigma=1$, $q=0$, $a=0$, $b=L$, and $\phi(0)=\phi(L)=0$.

Note: $\left[-p\phi\frac{d\phi}{dx}\right]_a^b = \left[-\phi\frac{d\phi}{dx}\right]_0^L = -\cancel{\phi(L)}^{0}\frac{d\phi}{dx}(L) + \cancel{\phi(0)}^{0}\frac{d\phi}{dx}(0) = 0$

So RQ becomes:
$$\boxed{\lambda = \frac{\int_0^L\left(\frac{d\phi}{dx}\right)^2dx}{\int_0^L \phi^2\,dx}}$$

We can see that all eigenvalues must be non-negative.

Furthermore, $\lambda=0$ can only occur if $\frac{d\phi}{dx}=0$, which means ϕ is constant. The only constant solution that satisfies the boundary values is the trivial solution, which is not an eigenfunction.

Thus, all eigenvalues are positive.

We know: $\phi_n(x)=\sin\left(\frac{n\pi}{L}x\right)$. So:
$$\lambda_n = \frac{\int_0^L\left(\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)\right)^2dx}{\int_0^L\left(\sin\left(\frac{n\pi}{L}x\right)\right)^2dx} = \left(\frac{n\pi}{L}\right)^2\frac{\int_0^L\cos^2\left(\frac{n\pi}{L}x\right)dx}{\int_0^L\sin^2\left(\frac{n\pi}{L}x\right)dx}$$

Thus, $\lambda_n=\left(\frac{n\pi}{L}\right)^2$, which is true.

WORKSHEET:

1. $$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \underbrace{\phi(-L) = \phi(L) \quad \text{and} \quad \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)}_{\text{These are not regular S-L boundary conditions.}}$$

1, 2: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \ldots$

3: There is **not** a unique eigenfunction per eigenvalue.

For $n > 0$, $\sin\left(\frac{n\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$ are eigenfunctions

4: $f \sim \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$

5: Orthogonality of eigenfunctions holds

2. $$\phi''(x) + 4\phi'(x) + 8\phi(x) + \lambda\phi(x) = 0, \qquad \phi(0) = \phi(L) = 0$$

(a) Multiply by e^{4x}:
$$\underbrace{e^{4x}\phi'' + 4e^{4x}\phi'}_{\text{product rule!}} + 8e^{4x}\phi + \lambda e^{4x}\phi = 0$$

$$\frac{d}{dx}\left[e^{4x}\phi'\right] + 8e^{4x}\phi + \lambda e^{4x}\phi = 0$$

We have regular S-L form with $p(x) = e^{4x}$, $q(x) = 8e^{4x}$, $\sigma(x) = e^{4x}$.

(b) Solve: $\phi'' + 4\phi + (8+\lambda)\phi = 0$

Characteristic polynomial: $r^2 + 4r + (8+\lambda) = 0$

So $r = \dfrac{-4 \pm \sqrt{4^2 - 4(8+\lambda)}}{2} = -2 \pm \sqrt{4 - (8+\lambda)} = -2 \pm \sqrt{-\lambda - 4}$

- If $-\lambda - 4 > 0$: $\phi(x) = Ae^{(-2+\sqrt{-\lambda-4})x} + Be^{(-2-\sqrt{-\lambda-4})x}$

  Boundary: $\phi(0) = \phi(L) = 0$

  $\phi(0) = A + B = 0$ so $A = -B$

  Consider $\phi(L)$ and conclude there are no nontrivial solutions

- If $-\lambda - 4 = 0$: $\phi(x) = Ae^{-2x} + Bxe^{-2x} \Rightarrow$ Again, no nontrivial solutions

To be continued...