LAST TIME:

$$r^2 \frac{d^2f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - m^2) f = 0 \qquad 0 \le r \le a$$

divide by r to put into S-L form, $p(r) = r$, $q(r) = \frac{-m^2}{r}$, $\sigma(r) = r$

Let $z = \sqrt{\lambda}\, r$:

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0$$

Bessel's eq. of order m

Solutions: Bessel functions $J_m(z)$ (first kind)

$Y_m(z)$ (second kind)

(d) $f(z) = c_1 J_m(z) + c_2 Y_m(z)$

$|f(0)| < \infty$ implies $c_2 = 0$, so $f(z) = c_1 J_m(z)$

$f(a) = 0$ implies $c_1 J_m(a) = 0$ we need a to be a zero of J_m

So: let z_{mn} be the n^{th} zero of J_m

$a \in \{z_{m1}, z_{m2}, z_{m3}, \dots\}$, and $z = \sqrt{\lambda}\, r$, or $\lambda = \left(\frac{z}{r}\right)^2$

eigenvalues: $\lambda = \left(\frac{z_{mn}}{a}\right)^2$ evaluating at $r = a$

(e) Orthogonality:

$$\int_0^a J_m\Big(\underbrace{\frac{z_{mp}}{a}}_{\text{eigenvalue}} r\Big)\, J_m\Big(\underbrace{\frac{z_{mq}}{a}}\, r\Big)\, \underbrace{r}_{\text{weight}}\, dr$$

Vibrating Drum Solution:

$$u(r,\theta,t) = \sum_{\substack{m \ge 0 \\ n \ge 0}} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \begin{Bmatrix} \cos(ct\sqrt{\lambda_{mn}}) \\ \sin(ct\sqrt{\lambda_{mn}}) \end{Bmatrix}$$

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

FINITE DIFFERENCE APPROXIMATIONS

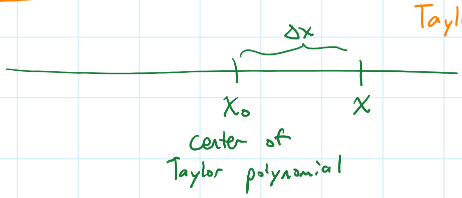
How can we approximate a function with a polynomial? **Taylor series!**

TAYLOR'S THEOREM: If f has $k+1$ derivatives at $x = x_0$, then

k^{th} deriv. of f

$$f(x_0+\Delta x) = \underbrace{f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(\Delta x)^k}_{\text{Taylor polynomial of degree } k} + \underbrace{\frac{f^{(k+1)}(\xi_{k+1})}{(k+1)!}(\Delta x)^{k+1}}_{\text{error term}}$$

error term, $\xi_{k+1} \in (x_0, x_0+\Delta x)$



EXAMPLE: First-degree Taylor polynomial

$$f(x_0+\Delta x) = \underbrace{f(x_0) + f'(x_0)\Delta x}_{\text{approx. for } f(x_0+\Delta x)} + \frac{f''(\xi_2)}{2}(\Delta x)^2$$

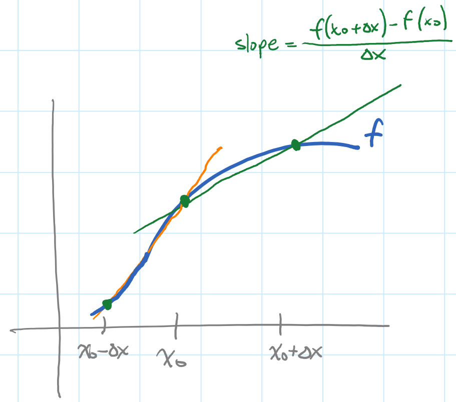
$$f(x_0+\Delta x) \approx f(x_0) + \underbrace{f'(x_0)}_{\text{solve for this}}\Delta x$$

error

FORWARD DIFFERENCE APPROX. FOR f'

$$f'(x_0) \approx \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - \frac{f''(\xi_2)}{2}\Delta x$$

difference quotient from calculus class



Truncation error $R = \left|\frac{f''(\xi_2)}{2}\Delta x\right| \le C\,|\Delta x|$ where $C = \max_{x\in(x_0,\,x_0+\Delta x)} \frac{1}{2}f''(x)$

$$R \le C\,|\Delta x| \qquad R = O(\Delta x)$$

First-order approx. error is (at worst) proportional to the first power of Δx.

DEF: $\underbrace{R = O(\Delta x^n)}$ means that $\lim_{\Delta x\to 0} \frac{R}{\Delta x^n} = \text{constant} < \infty$

"R is big-O of Δx^n"

EXAMPLE: $5\Delta x^2 + 1000\,\Delta x^3 = O(\Delta x^2)$

$$\lim_{\Delta x\to 0} \frac{5\Delta x^2 + 1000\,\Delta x^3}{\Delta x^3} = \infty$$

$$\lim_{\Delta x\to 0} \frac{5\Delta x^2 + 1000\,\Delta x^3}{\Delta x^2} = 5$$

BACKWARD DIFFERENCE APPROX OF f': $f'(x_0) \approx \frac{f(x_0-\Delta x) - f(x_0)}{-\Delta x}$, also has error $O(\Delta x)$

WORKSHEET: DERIVATIVE APPROXIMATIONS

1. (a) $f(x_0+\Delta x) = f(x_0) + f'(x_0)\,\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 + O(\Delta x^3)$

   (b) $f(x_0-\Delta x) = f(x_0) - f'(x_0)\,\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 + O(\Delta x^3)$

   (c) $f(x_0+\Delta x) - f(x_0-\Delta x) = 2f'(x_0)\,\Delta x + O(\Delta x^3)$

   $\frac{O(\Delta x^3)}{\Delta x}$

   Thus: $$\underbrace{f'(x_0) = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}}_{\text{CENTERED DIFFERENCE APPROX. FOR } f'} + O(\Delta x^2)$$