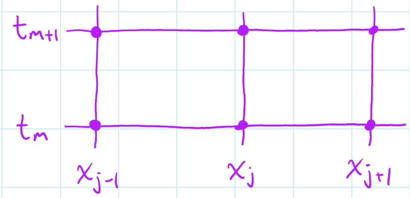
CRANK-NICOLSON SCHEME

(for the heat equation with zero boundary conditions)

STENCIL:



FINITE DIFFRENCE EQUATION

$$-\frac{s}{2}u_{j-1}^{(m+1)} + (1+s)u_j^{(m+1)} - \frac{s}{2}u_{j+1}^{(m+1)} = \frac{s}{2}u_{j-1}^{(m)} + (1-s)u_j^{(m)} + \frac{s}{2}u_{j+1}^{(m)}$$

IMPLEMENTATION

Matrix form: $A u^{(m+1)} = B u^{(m)}$

where

$$A = \begin{bmatrix} 1+s & -\frac{s}{2} & 0 & \cdots \\ -\frac{s}{2} & 1+s & -\frac{s}{2} & \cdots \\ 0 & -\frac{s}{2} & 1+s & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1-s & \frac{s}{2} & 0 & \cdots \\ \frac{s}{2} & 1-s & \frac{s}{2} & \cdots \\ 0 & \frac{s}{2} & 1-s & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Try it in Mathematica!

STABILITY:

Let $s = \frac{k\Delta t}{\Delta x^2}$ and $u_j^{(m)} = e^{i\alpha x_j} Q^m$.

Then:

$$e^{i\alpha x_j}Q^{m+1} - e^{i\alpha x_j}Q^m = \frac{s}{2}\Big(e^{i\alpha(x_j+\Delta x)}Q^m - 2e^{i\alpha x_j}Q^m + e^{i\alpha(x_j-\Delta x)}Q^m$$
$$+ e^{i\alpha(x_j+\Delta x)}Q^{m+1} - 2e^{i\alpha x_j}Q^{m+1} + e^{i\alpha(x_j-\Delta x)}Q^{m+1}\Big)$$

$$\cancel{e^{i\alpha x_j}}\cancel{Q^m}(Q-1) = \frac{s}{2}\cancel{e^{i\alpha x_j}}\cancel{Q^m}\left(e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x} + e^{i\alpha\Delta x}Q - 2Q + e^{-i\alpha\Delta x}Q\right)$$

$$Q - 1 = \frac{s}{2}\Big(2\cos(\alpha\Delta x) - 2 + Q\big(2\cos(\alpha\Delta x) - 2\big)\Big)$$

$$Q - 1 = s\Big(\cos(\alpha\Delta x) - 1 + Q\big(\cos(\alpha\Delta x) - 1\big)\Big)$$

Solve for Q:

$$Q - Qs\big(\cos(\alpha\Delta x) - 1\big) = 1 + s\big(\cos(\alpha\Delta x) - 1\big)$$

$$Q(1 + sw) = 1 - sw$$

Let $w = 1 - \cos(\alpha\Delta x)$.

Note $w \geq 0$.

Thus:

$$Q = \frac{1 - sw}{1 + sw}$$

If $sw \geq 0$, then this fraction is always between -1 and 1.

Since $s > 0$, we see that $|Q| < 1$ for all s, and so the Crank-Nicolson scheme is unconditionally stable.

FINITE DIFFERENCES FOR THE WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \quad t \ge 0$$

Boundary conditions: $u(0,t) = \alpha(t), \quad u(L,t) = \beta(t)$

Initial conditions: $u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(t)$

1. Centered differences for space and time:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$O(\Delta t^2) + \frac{u_j^{(m-1)} - 2u_j^{(m)} + u_j^{(m+1)}}{\Delta t^2} = c^2 \frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{\Delta x^2} + O(\Delta x^2)$$

Truncation error: $O(\Delta t^2) + O(\Delta x^2) = O(\Delta t^2 + \Delta x^2)$

2. Write $u_j^{(m+1)}$ in terms of u at previous time steps:

$$u_j^{(m+1)} = s^2 u_{j-1}^{(m)} + 2(1-s^2)u_j^{(m)} + s^2 u_{j+1}^{(m)} - u_j^{(m-1)}, \qquad \text{where } s = \frac{c\Delta t}{\Delta x}.$$

3. Matrix form: $U^{(m+1)} = AU^{(m)} - U^{(m-1)}$,

where $U^{(m)}$ is a vector of the computed values $u_1^{(m)}, u_2^{(m)}, \dots, u_{N-1}^{(m)}$.

$$\begin{bmatrix} u_1^{(m+1)} \\ \vdots \\ u_{N-1}^{(m+1)} \end{bmatrix} = \begin{bmatrix} 2(1-s^2) & s^2 & 0 & \cdots \\ s^2 & 2(1-s^2) & s^2 & \cdots \\ 0 & s^2 & 2(1-s^2) & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_1^{(m)} \\ \vdots \\ u_{N-1}^{(m)} \end{bmatrix} - \begin{bmatrix} u_1^{(m-1)} \\ \vdots \\ u_{N-1}^{(m-1)} \end{bmatrix}$$

4. (a) The forward difference can compute $U^{(1)}$ in terms of $U^{(0)}$, but with error $O(\Delta t)$.

However, we would prefer error not worse than $O(\Delta t^2)$.

(b) Centered difference is $O(\Delta t^2)$.

Approximation: $\frac{\partial u}{\partial t}(x_j, 0) \approx \frac{u_j^{(1)} - u_j^{(-1)}}{2\Delta t}$ or $2\Delta t \underbrace{\frac{\partial u}{\partial t}(x_j, 0)} \approx u_j^{(1)} - u_j^{(-1)}$

This is given by the initial condition $g(x)$.

Thus: $2\Delta t\, g(x_j) + u_j^{(-1)} = s^2 u_{j-1}^{(0)} + 2(1-s^2)u_j^{(0)} + s^2 u_{j+1}^{(0)} - u_j^{(-1)}$

$$u_j^{(-1)} = \frac{1}{2}\underbrace{\left[s^2 u_{j-1}^{(0)} + 2(1-s^2)u_j^{(0)} + s^2 u_{j+1}^{(0)} - 2\Delta t\, g(x_j)\right]}_{\text{all known quantities!}}$$

5. Stability analysis: Let $u_j^{(m)} = e^{i\alpha x_j} Q^m$.

$$e^{i\alpha x_j} Q^{m+1} = s^2 e^{i\alpha(x_j - \Delta x)} Q^m + 2(1-s^2) e^{i\alpha x_j} Q^m + s^2 e^{i\alpha(x_j+\Delta x)} Q^m - e^{i\alpha x_j} Q^{m-1}$$

$$Q^2 = s^2 Q e^{i\alpha\Delta x} + 2(1-s^2)Q + s^2 Q e^{-i\alpha\Delta x} - 1$$

$$Q^2 = Q\left[2s^2\cos(\alpha\Delta x) + 2 - 2s^2\right] - 1$$

$$Q^2 = Q\left[2 + 2s^2(\cos(\alpha\Delta x) - 1)\right] - 1$$

$2\sin^2\theta = 1 - \cos(2\theta)$

$$Q^2 = Q\left[2 - 4s^2\sin^2\left(\frac{\alpha\Delta x}{2}\right)\right] - 1$$

6. Quadratic formula: Let $\sigma = 1 - 2s^2\sin^2\left(\frac{\alpha\Delta x}{2}\right)$.

Then $Q^2 - 2\sigma Q + 1 = 0$, so

$$Q = \sigma \pm \sqrt{\sigma^2 - 1}$$

If $|\sigma| < 1$, then Q is complex: $Q = \sigma \pm i\sqrt{1-\sigma^2}$.

Then $|Q| = \sqrt{\sigma^2 + (1-\sigma^2)} = 1$, so the numerical scheme is stable.

If $|\sigma| = 1$, then $Q = \sigma = \pm 1$, and the scheme is stable.

If $|\sigma| > 1$, then Q is real.

Note that $\sigma = 1 - 2s^2\sin^2\left(\frac{\alpha\Delta x}{2}\right) \le 1$, so consider $\sigma < -1$.

If $\sigma < -1$, then $Q_- = \sigma - \sqrt{\sigma^2 - 1} < -1$, so the scheme is unstable.

Thus, this numerical scheme is stable iff $|\sigma| \le 1$. That is:

$$-1 \le 1 - 2s^2\sin^2\left(\frac{\alpha\Delta x}{2}\right) \le 1$$

This requires that $0 \le s^2 \le 1$, which implies $\underbrace{s = \frac{c\,\Delta t}{\Delta x} \le 1}_{\text{"Courant Stability Condition" for the wave equation}}$.