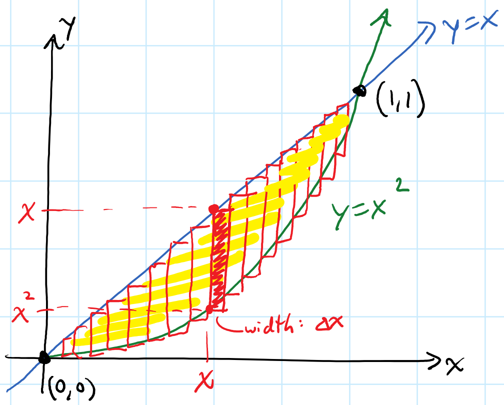
Wolfram Alpha and Desmos

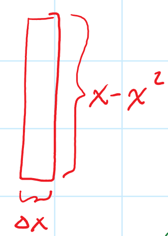
Great resources for computation

Cautions:

- Respond to what you type, not what you thought you typed.
- Might return mysterious results.
- Do not replace understanding.

COMPUTING AREA BETWEEN TWO FUNCTIONS





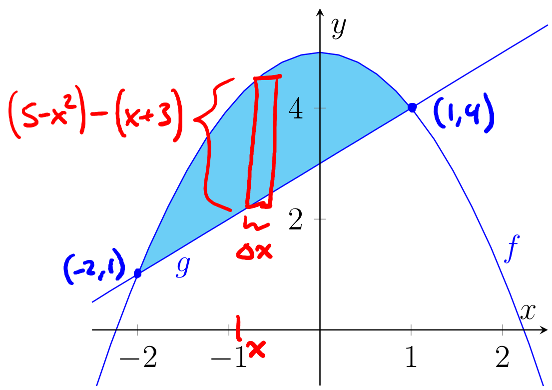
"Area Element"

$\Delta A = (x - x^2)\,\Delta x$

Total Area: $A = \int dA = \int_0^1 (x - x^2)\,dx$

Area ~~and Length~~

1. Use the following steps to find the area enclosed by the graphs of $f(x) = 5 - x^2$ and $g(x) = x + 3$, illustrated here.



(a) Identify the intersection points of f and g.

$5 - x^2 = x + 3$ has solutions $x = -2$, $x = 1$

(b) Draw rectangles of width Δx that approximate the area between the graphs. Write a formula for the area of such a rectangle in terms of x and Δx. Call the area of your rectangle ΔA, your *area element*.

$$\Delta A = \Big((5-x^2)-(x+3)\Big)\Delta x = (2-x-x^2)\Delta x$$

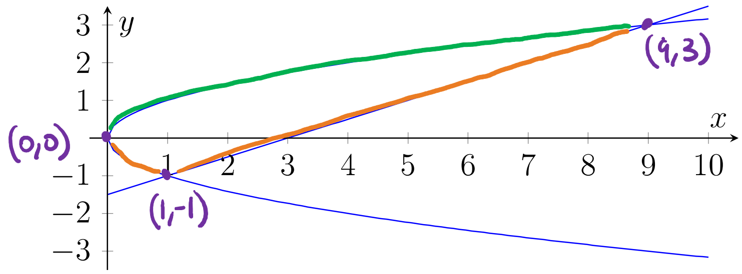
(c) Use the formula you found in (b) to write an integral that represents the area between the graphs.

$$\text{Area} = \int_{-2}^{1} (2-x-x^2)\,dx$$

(d) Evaluate the integral you wrote in (c) to find the area between the graphs.

$$\text{Area} = \int_{-2}^{1} (2-x-x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \frac{9}{2}$$

2. Here is a graph of three functions: $y = \sqrt{x}$, $y = -\sqrt{x}$, and $y = \frac{1}{2}x - \frac{3}{2}$. We want the find the area enclosed by all three graphs.



(a) What are the intersection points that we need to find?

$\sqrt{x} = \frac{1}{2}x - \frac{3}{2}$ $\quad x=9$

$-\sqrt{x} = \frac{1}{2}x - \frac{3}{2}$ $\quad x=1$

$\sqrt{x} = -\sqrt{x}$ $\quad x=0$

(b) Which function forms the "top" boundary of the region, and which function forms the "bottom" boundary?

"top": $\sqrt{x}$

"bottom": $-\sqrt{x}$ from 0 to 1

$\frac{1}{2}x - \frac{3}{2}$ from 1 to 9

(c) Write an integral expression that represents the area of the region.

Hint: You will need TWO different integrals.

Area: $A = \int_0^1 \left(\sqrt{x} - (-\sqrt{x})\right)dx + \int_1^9 \left(\sqrt{x} - \left(\frac{1}{2}x - \frac{3}{2}\right)\right)dx$

3. The previous problem will be easier if we change our perspective. Combine the equations $y = \sqrt{x}$ and $y = -\sqrt{x}$ into the single equation $x = y^2$.

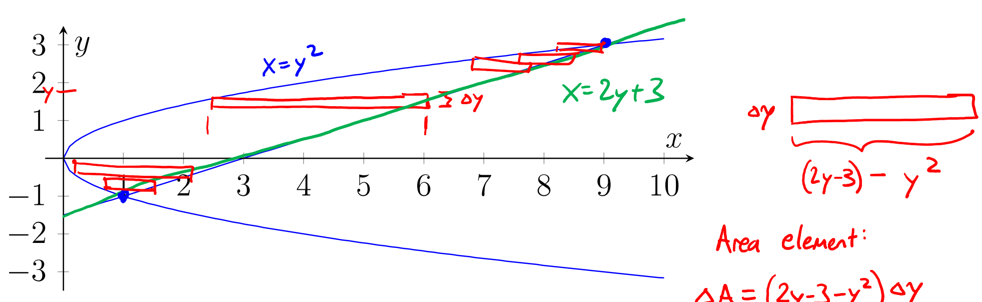
(a) Now write the other equation as $x =$ something.

$y = \frac{1}{2}x - \frac{3}{2}$ $\quad x = 2y + 3$

(b) Find the y-values of the intersection points.

$y = -1, \; y = 3$

(c) Draw *horizontal* rectangles (of height Δy that approximate the area between the functions. Write a formula for the area of such rectangles in terms of y and Δy.



Area element:

$\Delta A = (2y - 3 - y^2)\Delta y$

(d) Write a single integral (using the variable y) that represents the desired area.

Area: $A = \int_{-1}^{3} (2y - 3 - y^2)\, dy$

The arc length of the graph of a differentiable function $f(x)$ on an interval $[a,b]$ is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

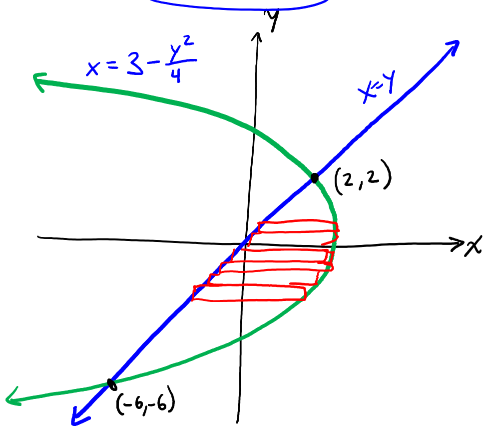
4. Find the length of the portion of the parabola $y = 4x - x^2$ that is above the x-axis. You may use technology to evaluate the integral.

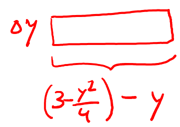
We'll do this on Monday.

5. Find the length of the graph of $y = x + \cos(x)$ for $0 \leq x \leq \pi$. Use technology to evaluate the integral.

... Monday

6. Find the area between the line $y = x$ and the parabola $4x + y^2 = 12$. Is this easier using vertical or horizontal rectangles?





Area Element:

$$\Delta A = \left(3 - \frac{y^2}{4} - y\right)\Delta y$$

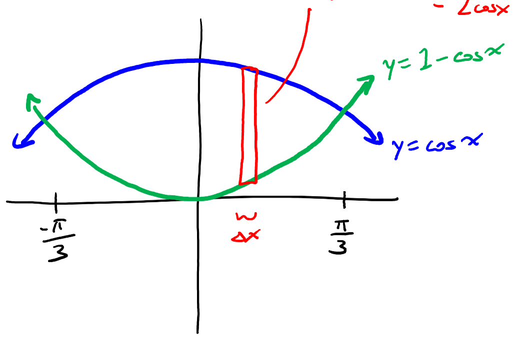
Area:

$$A = \int_{-6}^{2} \left(3 - \frac{y^2}{4} - y\right) dy$$

$$= \left[3y - \frac{y^3}{12} - \frac{y^2}{2}\right]_{-6}^{2} = \frac{64}{3} \approx 21.333$$

7. Find the area between the curves $y = \cos(x)$ and $y = 1 - \cos(x)$, for $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

height: $\cos x - (1 - \cos x)$
$= 2\cos x - 1$



Area Element:

$$\Delta A = (2\cos x - 1)\,\Delta x$$

Area: $A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos x - 1)\, dx = \Big[2\sin(x) - x\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$$= \left(2\sin\left(\tfrac{\pi}{3}\right) - \tfrac{\pi}{3}\right) - \left(2\sin\left(-\tfrac{\pi}{3}\right) + \tfrac{\pi}{3}\right)$$

$$= 2\left(\tfrac{\sqrt{3}}{2}\right) - \tfrac{\pi}{3} - 2\left(\tfrac{-\sqrt{3}}{2} - \tfrac{\pi}{3}\right) = \boxed{2\sqrt{3} - \tfrac{2\pi}{3} \approx 1.37}$$

8. (a) Write an integral that represents length of the graph of $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$.

... Monday

(b) Show that $1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2 = \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$

(c) Use part (b) to evaluate the integral you wrote in part (a).