Derivative Review — SOLUTIONS

Work with the people at your table to see how much you can remember about derivatives!

1. The **derivative** of a function at a point represents the slope of the tangent line at that point. That is:

$$f'(a) = \text{“the slope of the tangent line to the graph of } f(x) \text{ at } x = a\text{”}$$

2. Because of #1, the derivative can be used to understand the graph of a function $f(x)$:

   (a) If $f'(x) > 0$, then the graph of $f(x)$ is increasing.

   (b) If $f'(x) < 0$, then the graph of $f(x)$ is decreasing.

   (c) If $f'(x) = 0$, then the graph of $f(x)$ ~~is~~ has a horizontal tangent.

3. What happens at points where the derivative changes sign?

   (a) If f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$.

   (b) If f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$.

4. The **second derivative** of a function at a point represents the concavity of the graph at that point. That is:

$$f''(a) = \text{“the concavity of the graph of } f(x) \text{ at } x = a\text{”}$$

5. Because of #4, the second derivative can be used to understand the graph of a function $f(x)$:

   (a) If $f''(x) > 0$, then the graph of $f(x)$ is concave up.

   (b) If $f''(x) < 0$, then the graph of $f(x)$ is concave down

6. A point at which the graph of $f(x)$ changes concavity is called an inflection point.

   ⚠ **CAUTION:** Not every point at which $f''(x) = 0$ is one of these points!

7. Derivatives of specific kinds of functions:

   (a) **Power Rule:**
   If n is a real number and $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$.

   (b) **Trigonometric Functions**:
   If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

   If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

   If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$.

   (c) **Exponential Functions:**
   If $a > 0$ and $f(x) = a^x$, then $f'(x) = a^x \cdot \ln(a)$.

   In particular, if $f(x) = e^x$, then $f'(x) = e^x$.

   (d) **Natural Logarithm:**
   If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$.

8. Derivative "rules" for combining functions. You may assume that $f'(x)$ and $g'(x)$ both exist.

(a) **Constant Multiple Rule:**
If C is a constant, then $[C \cdot f(x)]' = \underline{C \cdot f'(x)}$.

(b) **Function Sum/Difference Rule:**

$$[f(x) + g(x)]' = \underline{f'(x) + g'(x)}.$$

$$[f(x) - g(x)]' = \underline{f'(x) - g'(x)}.$$

(c) **Product Rule:**

$$[f(x)g(x)]' = \underline{f'(x)g(x) + f(x)g'(x)}.$$

(d) **Chain Rule:** (composition of functions)

$$[f(g(x))]' = \underline{f'(g(x))\,g'(x)}.$$

(e) **Quotient Rule:**

$$\left[\frac{f(x)}{g(x)}\right]' = \underline{\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}.$$

9. Let's take some derivatives! Find f' and write down which method(s) you use for each:

(a) $f(x) = x + \sqrt{x} = x + x^{1/2}$

$$f'(x) = 1 + \frac{1}{2}x^{-1/2} = 1 + \frac{1}{2\sqrt{x}}$$ ← sum rule and power rule

(b) $f(x) = 2 + \frac{1}{x} + \frac{1}{x^2} = 2 + x^{-1} + x^{-2}$

$$f'(x) = -x^{-2} - 2x^{-3} = \frac{-1}{x^2} - \frac{2}{x^3}$$ ← sum rule and power rule

(c) $f(x) = \sqrt{x}\cos(x)$

$$f'(x) = \frac{1}{2\sqrt{x}}\cos(x) - \sqrt{x}\sin(x)$$ ← product rule, power rule, and derivative of $\cos(x)$

(d) $f(x) = \left[x^2 + \sin(x)\right]^4$

$$f'(x) = 4\left(x^2 + \sin(x)\right)^3\left(2x + \cos(x)\right)$$ ← power rule, chain rule, and derivative of $\sin(x)$

(e) $f(t) = e^{5+\ln t}$ $\xrightarrow{\text{alternate approach}}$ $f(t) = e^5 e^{\ln t} = e^5 \cdot t$

so $f'(t) = e^5$

$$f'(t) = e^{5+\ln t}\left(\frac{1}{t}\right)$$ ← chain rule and derivatives of e^t and $\log t$

NOTE: $e^{5+\ln t}\left(\frac{1}{t}\right) = e^5 e^{\ln t}\left(\frac{1}{t}\right) = e^5 t\left(\frac{1}{t}\right) = e^5$

(f) $f(x) = 3\sqrt{x}\cos(x^2) + \pi$

$$f'(x) = \frac{3}{2\sqrt{x}}\cos(x^2) + 3\sqrt{x}\left(-\sin(x^2)(2x)\right)$$ ← product rule, power rule, derivative of $\cos(x)$, and chain rule