The Substitution Rule

1. Review: Find the derivative of $f(x) = e^{4x}$. ☜ Chain rule!

$f'(x) = 4e^{4x}$

Then evaluate: $\int 4e^{4x}\,dx = e^{4x} + C$ $\qquad$ $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$ ☜ Hint: exactly what is different here?

NOTE: $\frac{d}{dx}\left(\frac{1}{4}e^{4x}\right) = \frac{1}{4}\cdot 4e^{4x} = e^{4x}$

2. Review: Find the derivative of $f(x) = \sin(x^2)$. Now evaluate the following:

$f'(x) = 2x\cos(x^2)$

$\int 2x\cos(x^2)\,dx$

$= \sin(x^2) + C$

$\int 5x\cos(x^2)\,dx$

$= \frac{5}{2}\int 2x\cos(x^2)\,dx$

$= \frac{5}{2}\sin(x^2) + C$

$\int 3x^2\cos(x^3)\,dx$

$= \sin(x^3) + C$

Since $\frac{d}{dx}\left(\sin(x^3)\right) = 3x^2\cos(x^3)$

3. With your group, act out the following dialogue:

Lana: I really need to evaluate $\int \sin(x)e^{\cos(x)}\,dx$
Erez: OK. Let's try substitution for this one!
Lana: Why do you think substitution will work? We are supposed to find something to make u and then search for the derivative of u, also. Oh, I guess we can do that!
Erez: Yeeeeeeah! We can let u be $\sin(x)$, and I see its derivative $\frac{du}{dx} = \cos(x)$ is also in there. Easy!
Lana (shaking her head): Erez, Erez, Erez...

(a) Why is Lana shaking her head at Erez?

If $u = \sin(x)$, then $du = \cos(x)dx$. But the given integral doesn't contain $\cos(x)\,dx$.

(b) What would be a more useful choice for u? Why?

Try $u = \cos(x)$. Then $du = -\sin(x)\,dx$.

After explaining her idea to Erez, Lana and Erez are left with the problem $\int -e^u\,du$.

Lana: I can do that integral!! The final answer is $-e^u + C$. Yes!!
Erez (shaking his head): My turn! Lana, Lana, Lana...

(c) Why is Erez shaking his head at Lana?

The final answer should be in terms of x, not u.

That is: $\int -e^u\,du = -e^u + C = -e^{\cos(x)} + C$.

4. For each part, everyone should think alone for 20 seconds and not write anything. Then, as a group, have a quick chat about what you might choose for u and **why**. Finally, check to see if you can finish each integral.

Go through all for problems

(a) $\int e^x \cos(e^x)\, dx = \int \cos(u)\, du = \sin(u) + C = \sin(e^x) + C$

$u = e^x$ $\quad du = e^x dx$

CHECK YOUR ANSWERS BY DIFFERENTIATING!

(b) $2\int \frac{e^{\sqrt{x}} + 4}{2\sqrt{x}}\, dx = 2\int (e^u + 4)\, du = 2e^u + 8u + C = 2e^{\sqrt{x}} + 8\sqrt{x} + C$

$u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}} dx$

(c) $\int \sqrt{x-4}\, dx = \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(x-4)^{3/2} + C$

Hint: $\sqrt{x-4} = \sqrt{x-4} \cdot 1$

$u = x - 4$ $\quad du = 1\, dx$

(d) $\int xe^{x^2} \cos\left(e^{x^2}\right) dx = \int \cos(u)\, du = \sin(u) + C = \sin\left(e^{x^2}\right) + C$

$u = e^{x^2}$ $\quad du = 2xe^{x^2} dx$

5. The beauty of the substitution method for definite integrals is that you can also substitute for the bounds, meaning you *never* have to change back to the original variable. Find each of the following using u-substitution where you substitute u-values for each of the bounds.

(a) $\int_0^1 e^x \cos(e^x)\, dx = \int_1^e \cos(u)\, du = \sin(u)\Big|_{u=1}^{u=e} = \sin(e) - \sin(1)$

If $x=0$, then $u = e^0 = 1$. If $x = 1$, then $u = e^1 = e$.

(b) $\int_1^4 \frac{e^{\sqrt{x}} + 4}{\sqrt{x}}\, dx = 2\int_1^2 (e^u + 4)\, du = \left[2e^u + 8u\right]_{u=1}^{u=2} = (2e^2 + 16) - (2e + 8) = 2e^2 - 2e + 8$

If $x = 1$, then $u = \sqrt{1} = 1$. If $x = 4$, then $u = \sqrt{4} = 2$.

(c) $\int_5^8 \sqrt{x-4}\, dx = \int_1^4 \sqrt{u}\, du = \frac{2}{3}\left[u^{3/2}\right]_{u=1}^{u=4} = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3}$

If $x = 5$, then $u = 5 - 4 = 1$. If $x = 8$, then $u = 8 - 4 = 4$.

(d) $\int_0^1 xe^{x^2} \cos\left(e^{x^2}\right) dx = \int_1^e \cos(u)\, du = \sin(u)\Big|_{u=1}^{u=e} = \sin(e) - \sin(1)$

If $x = 0$, then $u = e^{0^2} = 1$. If $x = 1$, then $u = e^{1^2} = e$

6. If $\int_1^4 g(x)\, dx = 12$, evaluate $\int_0^1 g(3x+1)\, dx$.

$\frac{1}{3}\int_0^1 3g(3x+1)\, dx = \frac{1}{3}\boxed{\int_1^4 g(u)\, du} = \frac{1}{3}(12) = 4$

$u = 3x + 1$ $\quad du = 3dx$

7. More practice! Evaluate the following:

(a) $\int_0^3 2xe^{x^2-5}\,dx = \int_{-5}^{4} e^u\,du = e^u\Big|_{-5}^{4} = e^4 - e^{-5}$

$u = x^2-5 \qquad du = 2x\,dx$

(b) $\frac{1}{9}\int_0^1 \frac{9x^2}{2+3x^3}\,dx = \frac{1}{9}\int_2^5 \frac{1}{u}\,du = \frac{1}{9}\ln(u)\Big|_2^5 = \frac{1}{9}\left(\ln(5)-\ln(2)\right) = \frac{1}{9}\ln\left(\frac{5}{2}\right)$

$u = 2+3x^3 \qquad du = 9x^2\,dx$

(c) $\int \frac{e^{\ln x}}{x}\,dx = \int \frac{x}{x}\,dx = \int 1\,dx = x + C$

Is this simpler than it seems?

(d) $\frac{1}{2}\int \sin(2x)\,2\cos(2x)\,dx = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\left(\sin(2x)\right)^2 + C$

$u = \sin(2x) \qquad du = 2\cos(2x)\,dx$

(e) $-\frac{1}{2}\int_0^3 -2x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int_9^0 \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^0 = -\frac{1}{2}\left(\frac{2}{3}(0)^{3/2} - \frac{2}{3}(9)^{3/2}\right)$

$u = 9-x^2 \qquad du = -2x\,dx \qquad = -\frac{1}{2}\left(0 - \frac{2}{3}(27)\right) = 9$

(f) $\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

Hint: $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

$= \arctan(x) + \frac{1}{2}\int \frac{1}{u}\,du$

$= \arctan(x) + \frac{1}{2}\ln(u) + C$

$= \arctan(x) + \frac{1}{2}\ln(1+x^2) + C$