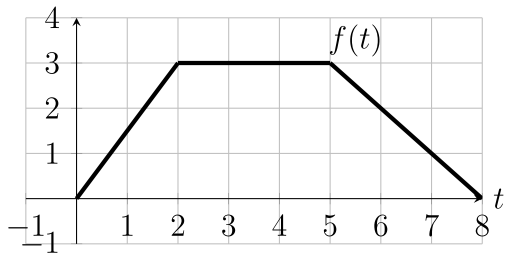
More Integration

1. The graph of a function $f(t)$ is shown below. Let $g(x) = \int_0^x f(t)\,dt$



(a) What are the following values of g?

$g(0) = 0$ $\qquad g(1) = \frac{3}{4}$ $\qquad g(2) = 3$ $\qquad g(3) = 6$

$g(4) = 9$ $\qquad g(5) = 12$ $\qquad g(6) = 14.5$ $\qquad g(7) = 16$

(b) **Felix:** Look! $g'(4) = 3$.

Group chat: Is Simon correct? Why or why not?

At $x=4$, it appears that g is increasing with a slope of 3, so $g'(4)=3$

Better answer: g accumulates area under f. At $x=4$, the rate of accumulation is 3 units of area for 1 unit of change in x.

(c) **Maura:** I need a value of x such that $g'(x) = 2$.

Group chat: Help Maura out. Where is $g'(x) = 2$?

$g'(x) = 2$ where $f(x)=2$, which is at $x = \frac{4}{3}$ and $x = 6$.

(d) **Simon:** Ah-ha! Now I see that f is the derivative of g!

2. Quick! Find the following derivatives!

$$\frac{d}{dx}\int_0^x t\sin(t)\,dt = x \cdot \sin(x)$$

$$\frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}$$

$$\frac{d}{dx}\int_x^0 |\cos(t)|\,dt = \frac{d}{dx}\left(-\int_0^x |\cos(t)|\,dt\right) = -|\cos(x)|$$

3. Now suppose we want to find $\frac{d}{dx}\int_1^{x^2} e^{t^2}\,dt$.

(a) Let $F(t)$ be an antiderivative of e^{t^2}. Express $\int_1^{x^2} e^{t^2}\,dt$ in terms of F.

$$\int_1^{x^2} e^{t^2}\,dt = F(x^2) - F(1) \qquad \text{where } F'(t) = e^{t^2}$$

(b) Differentiate what you wrote in part (a) and simplify. What have you found?

$$\frac{d}{dx}\int_1^{x^2} e^{t^2}\,dt = \frac{d}{dx}\left(F(x^2) - F(1)\right) = \underbrace{F'(x^2)(2x)}_{\text{by the chain rule}} - 0 = 2x\, e^{(x^2)^2} = 2x e^{x^4}$$

So: $\frac{d}{dx}\int_1^{x} e^{t^2}\,dt = 2x e^{x^4}$

4. If $F(x) = \int_0^{x^2} (t^2 - 10)\, dt$, what is $F'(x)$?

$F'(x) = \left((x^2)^2 - 10\right)(2x) = 2x(x^4 - 10)$

SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is a continuous function and c is any constant, then:

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x).$$

Also: $\frac{d}{dx}\int_c^{k(x)} f(t)\,dt = f(k(x))\,k'(x)$

(by the chain rule)

5. If $F(x) = \int_1^{\ln(x)} (s^6 + e^{4s})\, ds$, what is $F'(x)$?

$F'(x) = \left((\ln x)^6 + e^{4\ln x}\right) \cdot \frac{1}{x}$

6. If $h(x) = \int_{3x}^{0} \cos(y)\, dy$, what is $h'(x)$?

$h(x) = -\int_0^{3x} \cos(y)\,dy$, so $h'(x) = -\cos(3x)\cdot 3 = -3\cos(3x)$

7. If $q(x) = \int_{\ln(x)}^{2x} (1 + t^2)\, dt$, what is $q'(x)$?

$q(x) = \int_0^{2x}(1+t^2)\,dt - \int_0^{\ln(x)}(1+t^2)\,dt$

So: $q'(x) = \left(1 + (2x)^2\right)\cdot 2 - \left(1 + (\ln x)^2\right)\cdot\frac{1}{x} = 2(1+4x^2) - \frac{1}{x}\left(1 + (\ln x)^2\right)$

8. The *sine integral function*

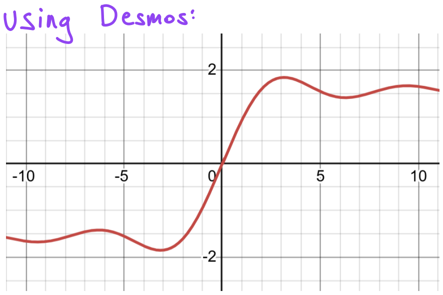
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

is important in electrical engineering.

(a) Use technology to sketch a graph of $\mathrm{Si}(x)$.

(b) At what values of x does $\mathrm{Si}(x)$ have local maximum values?

Using Desmos:



$\mathrm{Si}'(x) = \frac{\sin(x)}{x}$

Then: $\mathrm{Si}'(x) = 0$ where $\sin(x) = 0$

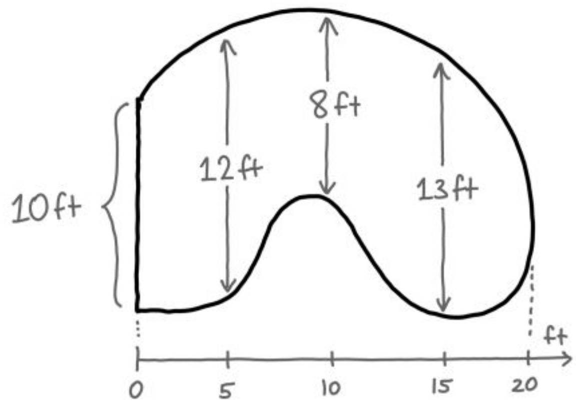
$\mathrm{Si}'(x)$ undefined at $x = 0$

So the critical points of $\mathrm{Si}(x)$ are $x = n\pi$ for integers n.

The first positive critical point is $x = \pi$.

Since $\mathrm{Si}'(x)$ is positive for x slightly less than π and negative for x slightly greater than π, $\mathrm{Si}(x)$ has a local max at $x = \pi$.

9. Chloe: I need to find the area of my backyard pool. I made a sketch of it and took some measurements:



Erez: I can't find the exact area of your pool, but I can find values M and N such that the area is *between* M and N.

Group chat: What is Erez talking about? Can you find such values M and N?

What is your *best estimate* of the area of the pool? How far off from the actual area could you be?

We can estimate the area of the pool using four rectangles.

A lower estimate can use the smaller of the left and right lengths for each rectangle:

$$M = 5(10 + 8 + 8 + 10) = 130 \text{ ft}^2.$$

An upper estimate can use the larger of the left and right lengths for each rectangle:

$$N = 5(12 + 12 + 13 + 13) = 250 \text{ ft}^2.$$

We could take the average of M and N as our best estimate:

$$\text{area} \approx \frac{M+N}{2} = \frac{130+250}{2} = \frac{380}{2} = 190 \text{ ft}^2,$$

and we know that this is off by no more than 60 ft^2

(since $250 - 190 = 60$ and $190 - 130 = 60$).

10. The speed of a runner was measured each second for the first five seconds of a race:

time t (seconds)	velocity v (meters/second)
0	0
1	6.8
2	8.2
3	9.8
4	10.2
5	10.4

(a) Approximately how far did the runner travel in these five seconds?

Left sum: $0 + 6.8 + 8.2 + 9.8 + 10.2 = 35$ m

Right sum: $6.8 + 8.2 + 9.8 + 10.2 + 10.4 = 45.4$ m

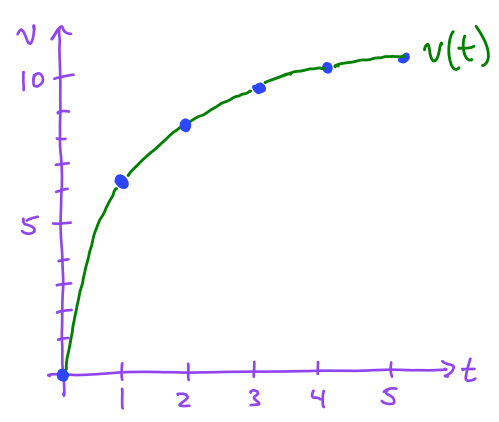
Average = trapezoid approximation $= \frac{35 + 45.4}{2} = 40.2$ m

(b) How far off could your answer in part (a) be from the actual distance that the runner traveled? Can you tell if your estimate is an overestimate or an underestimate?

We can't really tell without knowing more about the runner's speed.

(c) What additional assumptions could you make about the runner's velocity that would improve your answers in part (b)?

If we assume that the runner is accelerating (forward) throughout the five seconds, then we know that the velocity is always increasing, so a graph of $v(t)$ looks something like this:



This implies, for instance, that the left sum is an underestimate and the right sum is an overestimate.