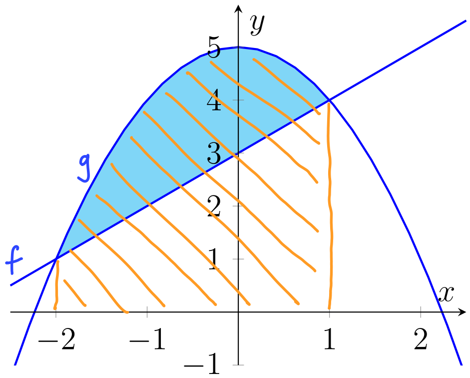
Area

1. The area enclosed by $f(x) = x + 3$ and $g(x) = 5 - x^2$ is shaded below.



Intersection points:

$x + 3 = 5 - x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or $x = 1$

Jade: I really want to find the area of the shaded region.

Sundar (bewildered): Why would you want to...oh, never mind. Well, first you need to find where the two functions intersect each other.

Jade (writing quickly): Well, I can do that. Clearly this is where $x = -2$ and where $x = 1$.

Sundar (bewildered): How did you find those?

Jade (ignoring Sundar's question): Got it: $\displaystyle\int_{-2}^{1} (5 - x^2)\, dx$.

Sundar (shaking his head): THAT'S TOO MUCH! You need to do more calculus.

(a) Why did Sundar say that it is important to find where the two functions intersect?

These x-values will be the bounds of integration.

(b) How did Jade actually find the two important values of x?

Set the two functions equal and solve for x.

Sorry, "look and the graph and guess" is not good enough.

(c) Shade the area that is really calculated when Jade finds $\displaystyle\int_{-2}^{1} (5 - x^2)\, dx$.

Shaded orange above.

(d) Write down an integral that *actually* computes the area of the region that Jade wants.

$$\int_{-2}^{1} \left((5-x^2) - (x+3) \right) dx = \int_{-2}^{1} (2 - x - x^2)\, dx = \left[2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_{-2}^{1} = \frac{9}{2}$$

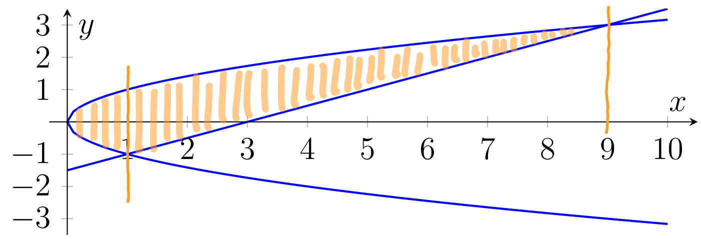
(e) **Group chat:** What is the "general strategy" to calculate the area *enclosed by* two functions $f(x)$ and $g(x)$?

Partition the area into slices of width Δx and height $g(x) - f(x)$, (assuming g is the "upper" function and f is the "lower" function).

The area of such a slice is $(g(x) - f(x))\Delta x$, which we call the area element

Integrate the area element between the bounds, replacing Δx with dx

2. Here is a graph of three functions: $y = \sqrt{x}$, $y = -\sqrt{x}$, and $y = \frac{1}{2}x - \frac{3}{2}$. We want the find the area enclosed by all three graphs.



(a) What are the intersection points that we need to find?

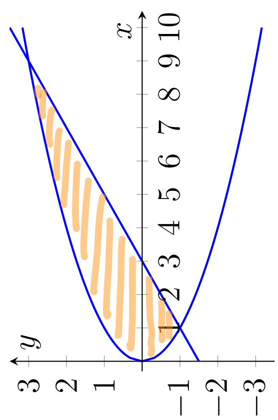
$\frac{1}{2}x - \frac{3}{2} = \sqrt{x}$ and $\frac{1}{2}x - \frac{3}{2} = -\sqrt{x}$

$(x = 9)$ $(x = 1)$

(b) Write an integral expression that represents the area of the region.

As you move from left to right, which function is "higher" and which is "lower"? Is one integral enough?

$$\int_0^1 \left(\sqrt{x} - (-\sqrt{x})\right) dx + \int_1^9 \left(\sqrt{x} - \left(\tfrac{1}{2}x - \tfrac{3}{2}\right)\right) dx$$



Delphine: What have you done to the picture?

Chris: I turned it 90 degrees! This will make the problem easier, I promise!

Delphine: Well, if we are going to look at it this way, we have to write the equations in a from $x =$ a function of y rather than $y =$ a function of x.

Chris: Ohhhhh...so $y = \sqrt{x}$ and $y = -\sqrt{x}$ both become $x = y^2$. Cool!

(c) Write the other equation in this new form.

$y = \frac{1}{2}x - \frac{3}{2} \xrightarrow{\text{solve for } x} x = 2y + 3$

(d) What are the "important values of y?

intersection points of $x = y^2$ and $x = 2y + 3$ $\longrightarrow$ $y^2 = 2y + 3$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

so $y = -1$ and $y = 3$

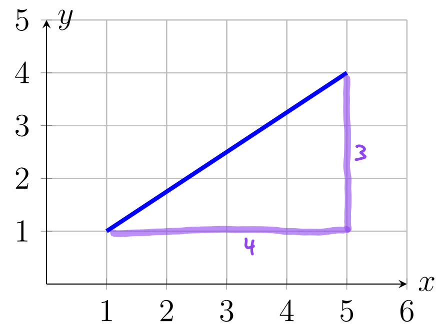
(e) Write a single integral (using the variable y) that represents the desired area.

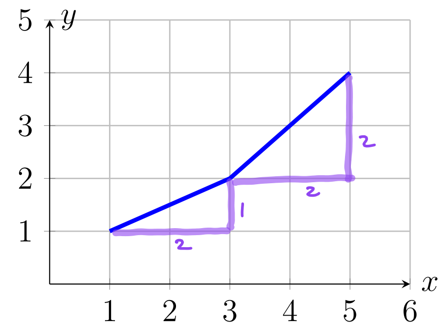
$$\int_{-1}^{3} \left(2y - 3 - y^2\right) dy$$

Length

1. Find the length of the line segment on the left. Then find the total length of the two line segments on the right.

length $= \sqrt{4^2+3^2}$
$= \sqrt{25}$
$= 5$





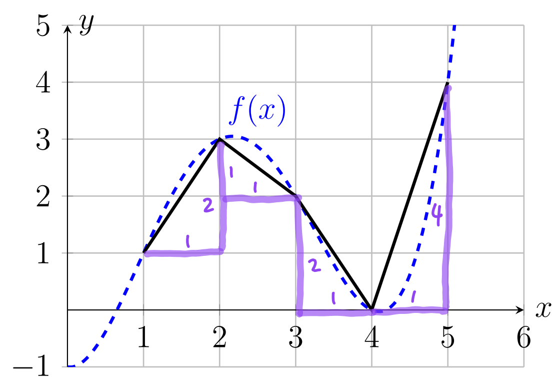
length $= \sqrt{2^2+1^2} + \sqrt{2^2+2^2}$
$= \sqrt{5} + \sqrt{8}$
≈ 5.064

2. Jade: Now I want to know the *length* of the graph of $f(x)$ between $x = 1$ and $x = 5$.

$$f(x) = \frac{5}{24}x^4 - \frac{7}{4}x^3 + \frac{91}{24}x^2 - \frac{1}{4}x - 1$$

Sundar: Wait, length? Not area? That seems awfully difficult to find exactly. What if we use the following picture to estimate it?

What are the solid lines for?



(a) **Group task:** Estimate the length that Jade wants.

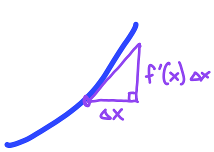
length $\approx \sqrt{1^2+2^2} + \sqrt{1^2+1^2} + \sqrt{1^2+2^2} + \sqrt{1^2+4^2} = \sqrt{5} + \sqrt{2} + \sqrt{5} + \sqrt{17} \approx 10.009$

(b) **Group chat:** How could you give a more accurate estimate?

Approximate the curve by more, shorter line segments.

(c) **Further group chat:** How could you get an *exact* answer?

If the line segments are really small, then they're basically tangent to $f(x)$.



The slope of a tangent segment is $f'(x)$.

The length of the tangent segment sketched at left is $\sqrt{\Delta x^2 + \Delta x^2 (f'(x))^2} = \underbrace{\sqrt{1+(f'(x))^2}\,\Delta x}_{\text{the length element}}$.

So we can find the exact length by integrating: $\int_a^b \sqrt{1+(f'(x))^2}\,dx$.

We didn't get to this page in class. We'll resume with arc length on Monday.

3. (a) Use the Pythagorean theorem to find the length of $y = 2x$ between $x = 0$ and $x = 2$.

(b) Check your answer to (a) by using the integral formula for arclength instead of the Pythagorean theorem.

4. Set up the integral that will find the arclength of $y = \frac{1}{2}x^2$ between $x = 0$ and $x = 2$.

5. Set up the integral that will find the arclength (that is, circumference) of the top half of a circle of radius 1.