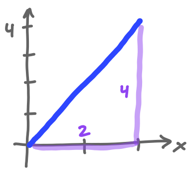
MATH 126 A/B • 22 September 2025

3. (a) Use the Pythagorean theorem to find the length of $y = 2x$ between $x = 0$ and $x = 2$.



$$\text{length} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \approx 4.472$$

(b) Check your answer to (a) by using the integral formula for arclength instead of the Pythagorean theorem.

$f(x) = 2x$, so $f'(x) = 2$

$$\text{length} = \int_0^2 \sqrt{1 + (f'(x))^2}\, dx = \int_0^2 \sqrt{1 + 2^2}\, dx = \int_0^2 \sqrt{5}\, dx = 2\sqrt{5} \approx 4.472$$

4. Set up the integral that will find the arclength of $y = \frac{1}{2}x^2$ between $x = 0$ and $x = 2$.

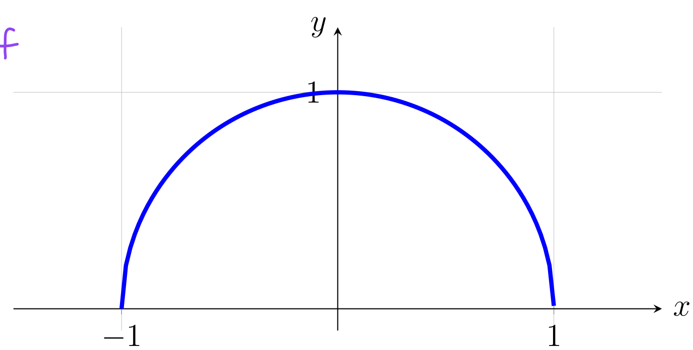
$y' = x$

so arclength is $\displaystyle\int_0^2 \sqrt{1 + x^2}\, dx$

5. Set up the integral that will find the arclength (that is, circumference) of the top half of a circle of radius 1.

Equation of a circle:

$x^2 + y^2 = 1$



Equation of the top half of a circle:

$f(x) = \sqrt{1 - x^2}$

Differentiate:

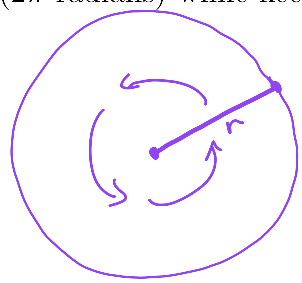
$$f'(x) = \tfrac{1}{2}(1 - x^2)^{-1/2}(2x)$$
$$= x(1 - x^2)^{-1/2}$$

$$\text{Arc length} = \int_{-1}^{1} \sqrt{1 + (f'(x))^2}\, dx = \int_{-1}^{1} \sqrt{1 + x^2(1 - x^2)^{-1}}\, dx$$

Volume

1. It's time to do some drawing!

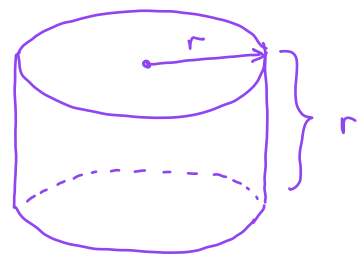
(a) Draw a single line segment. What shape do you get as you rotate the segment 360 degrees (2π radians) while keeping one end fixed? What is the area of this shape?



a circle

Area $= \pi r^2$

(b) Draw the best picture of a cylinder that you can. Label the cylinder so that the radius of the base is r and the height is h.

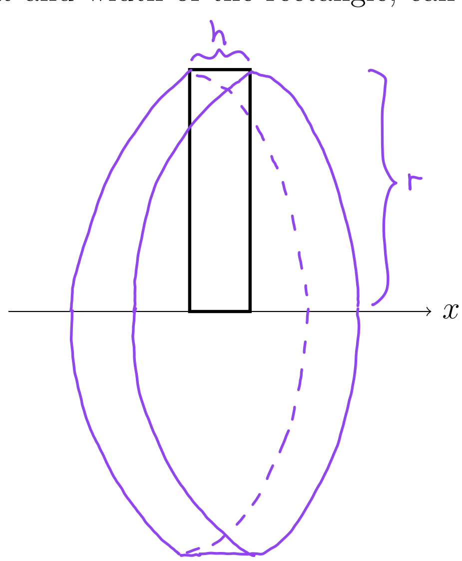


(c) Do you remember how to compute the *volume* of a cylinder? What is the formula? Try to explain to the people at your table *why* your formula works for the volume of a cylinder.

Volume $= \pi r^2 h$

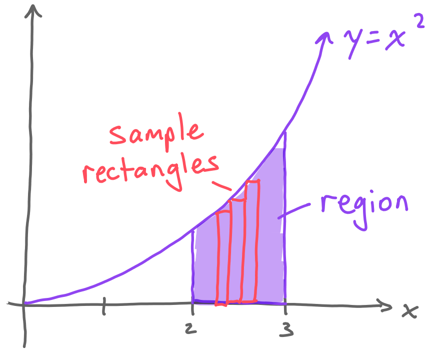
The volume of a cylinder is the area of the base multiplied by the height.

(d) Here is a rectangle. We want to rotate the rectangle 360 degrees around the x-axis. Imagine the rectangle coming towards you out of the page, then rotating around the x-axis to produce something 3D. Try drawing the resulting 3D shape. If you know the height and width of the rectangle, can you find the volume of your 3D shape?

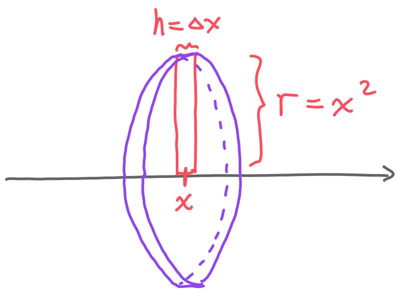


Volume $= \pi r^2 h$

2. (a) Carefully draw the region bounded by $f(x) = x^2$, $x = 2$, $x = 3$, and the x-axis. Draw some sample rectangles of the type that you would use to approximate the area of this region.



(b) Rotate one of your sample rectangles around the x-axis to get a 3D shape. What is the formula for the volume of this 3D shape?



Volume:

$$\Delta V = \pi \left(x^2\right)^2 \Delta x$$

(c) Now suppose the entire region from part (a) is rotated around the x-axis. What is the volume of the resulting shape?

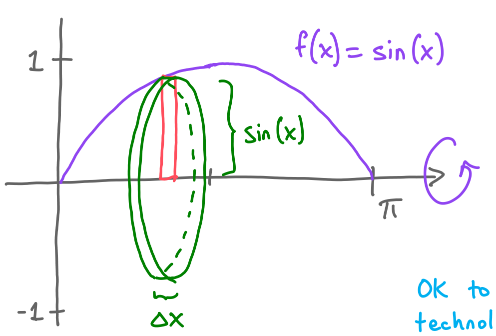
☞ Integrate the volume formula you found in part (b)!

$$\text{Volume} = \int_2^3 \pi \left(x^2\right)^2 dx = \pi \int_2^3 x^4 \, dx = \pi \cdot \tfrac{1}{5} x^5 \Big|_2^3$$

$$= \frac{\pi}{5}\left(3^5 - 2^5\right) = \frac{\pi}{5}\left(243 - 32\right) = \frac{211\pi}{5} \approx 132.58$$

3. Suppose the region between the graph of $f(x) = \sin(x)$ and the x-axis, for $0 \le x \le \pi$, is rotated around the x-axis. What is the volume of the resulting solid?

☞ Hint: $\sin^2(x) = \frac{1 - \cos(2x)}{2}$



Volume element: $\Delta V = \pi\left(\sin(x)\right)^2 \Delta x$

Volume of solid of revolution:

$$\boxed{V = \int_0^\pi \pi \sin^2(x)\, dx} = \pi \int_0^\pi \frac{1 - \cos(2x)}{2}\, dx$$

OK to use technology to evaluate this integral

$$= \frac{\pi}{2} \int_0^\pi \left(1 - \cos(2x)\right) dx = \frac{\pi}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_0^\pi$$

$$= \frac{\pi}{2}\left[\left(\pi - \tfrac{1}{2}\cancel{\sin(2\pi)}^{\,0}\right) - \left(0 - \tfrac{1}{2}\cancel{\sin(0)}^{\,0}\right)\right]$$

$$= \frac{\pi^2}{2} \approx 4.9348$$