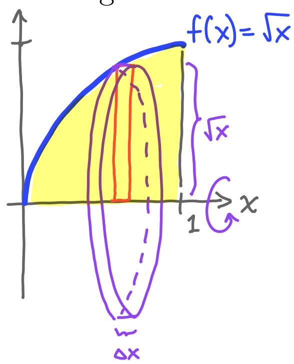
Volume

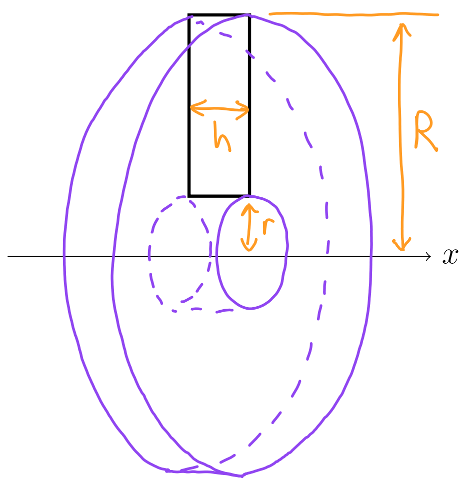
1. Sketch the region bounded by the graph of $f(x) = \sqrt{x}$ and the lines $x = 1$ and $y = 0$. If this region is rotated around the x-axis, what is the volume of the resulting solid?



Volume element: $\Delta V = \pi(\sqrt{x})^2 \Delta x = \pi x \,\Delta x$

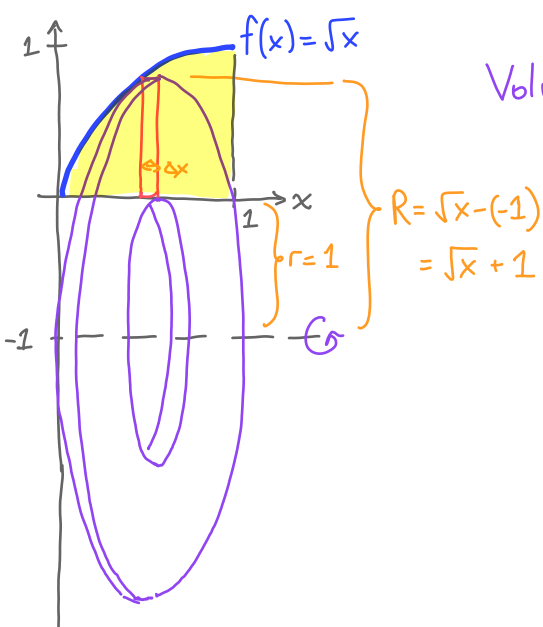
Volume: $V = \int_0^1 \pi x \,\Delta x = \frac{1}{2}\pi x^2 \Big|_0^1 = \boxed{\frac{\pi}{2}}$

2. Here is a rectangle. We want to rotate the rectangle 360 degrees around the x-axis. Try to sketch the resulting 3D shape. If you know the height and width of the rectangle and its distance from the x-axis, what is the volume of your 3D shape?



$$\text{Volume} = \pi R^2 h - \pi r^2 h$$
$$= \pi(R^2 - r^2)h$$

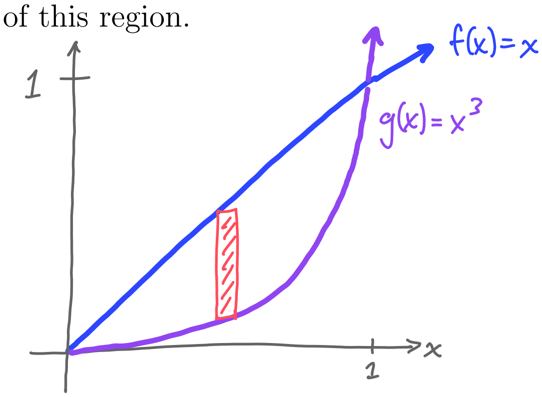
3. Now suppose the region from problem #1 is rotated around the line $y = -1$. Sketch the resulting 3D solid. When you slice this solid vertically, what is the shape of each slice? Write an integral that gives the volume of the 3D solid.



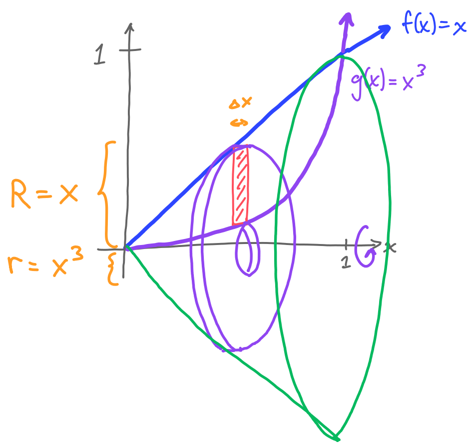
Volume element: $\Delta V = \pi\left((\sqrt{x}+1)^2 - (1)^2\right)\Delta x$

$$\text{Volume} = \int_0^1 \pi\left((\sqrt{x}+1)^2 - 1\right)dx$$
$$= \pi\int_0^1 (x + 2\sqrt{x})\,dx = \boxed{\frac{11}{6}\pi}$$

4. (a) Carefully draw the graph of $f(x) = x$, $g(x) = x^3$, and the region in the first quadrant bounded by both graphs. Draw a sample rectangle of the type that you would use to approximate the area of this region.



(b) Now suppose the entire region from part (a) is rotated around the x-axis. What is the volume of the resulting shape?



Volume element: $\Delta V = \pi\left(x^2 - (x^3)^2\right)\Delta x$

Rotate your rectangle from part (a) around the x-axis to obtain the volume element

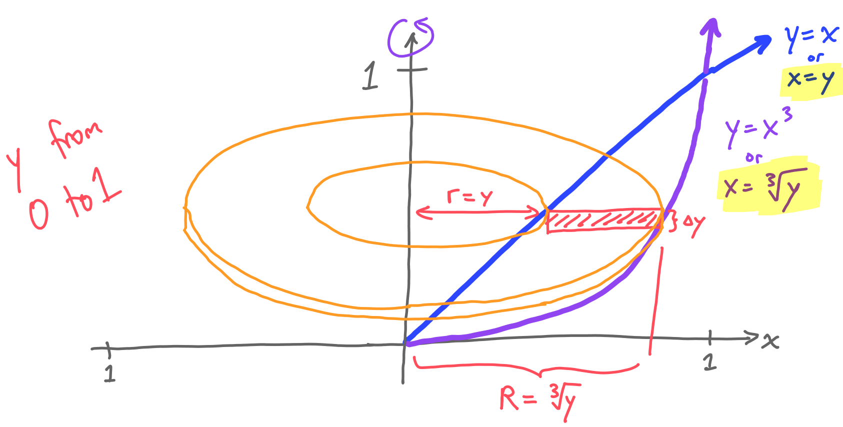
$$\text{Volume} = \int_0^1 \pi\left(x^2 - x^6\right)dx$$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{7}x^7\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \boxed{\frac{4}{21}\pi}$$

(c) **Chloe:** I want to rotate the region around the y-axis instead of the x-axis.

Milo: In that case, you will need to use a horizontal rectangle to create the volume element.

Group chat: What does Milo mean? Can you use his suggestion to find the volume of Chloe's 3D object?



Volume element:

$$\Delta V = \pi\left((\sqrt[3]{y})^2 - (y)^2\right)\Delta y$$

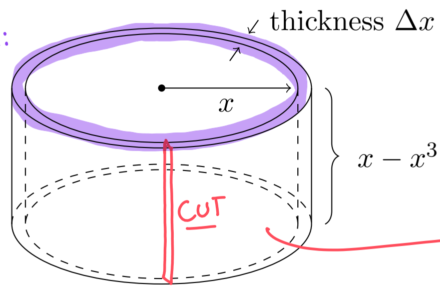
Volume: $V = \int_0^1 \pi\left(y^{2/3} - y^2\right)dy$

$$= \pi\left[\frac{3}{5}y^{5/3} - \frac{1}{3}y^3\right]_0^1$$

$$= \pi\left(\frac{3}{5} - \frac{1}{3}\right) = \boxed{\frac{4}{15}\pi}$$

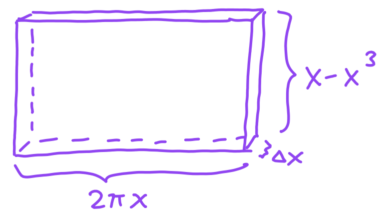
(d) **Marissa:** You *can* use a vertical rectangle for Chloe's object! You will just need volume elements that look like this:

circumference: $2\pi x$



This will not be on the exam.

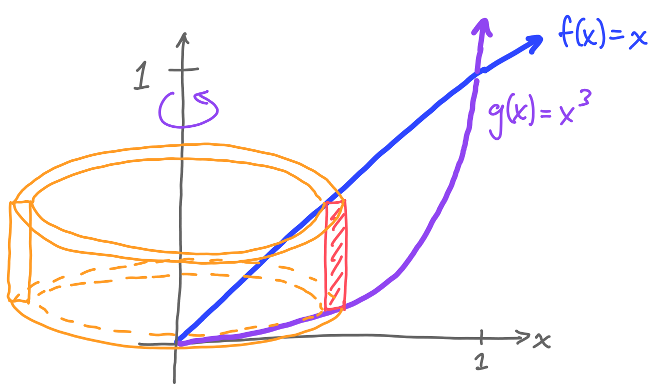
"Unroll" the circular tube:



thus, volume is $2\pi x(x-x^3)\Delta x$

Then $\Delta V = 2\pi x(x - x^3)\Delta x$.

Group chat: How did Marissa find her formula for the volume element? Check that this leads to the same volume as you found in part (c).



Volume element:

$$\Delta V = 2\pi x(x-x^3)\Delta x$$

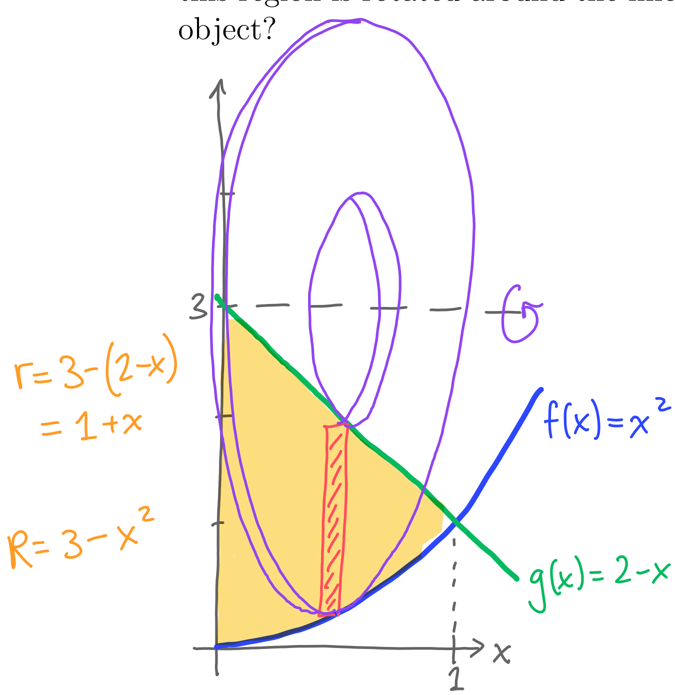
Volume:

$$V = \int_0^1 2\pi x(x-x^3)\,dx$$

$$= \int_0^1 2\pi(x^2-x^4)\,dx$$

$$= 2\pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \boxed{\tfrac{4}{15}\pi}$$

5. Sketch the region bounded by the graphs of $f(x) = x^2$, $g(x) = 2 - x$, and $x = 0$. If this region is rotated around the line $y = 3$, what is the volume of the resulting solid object?



Volume element:

$$\Delta V = \pi\left((3-x^2)^2 - (1+x)^2\right)\Delta x$$

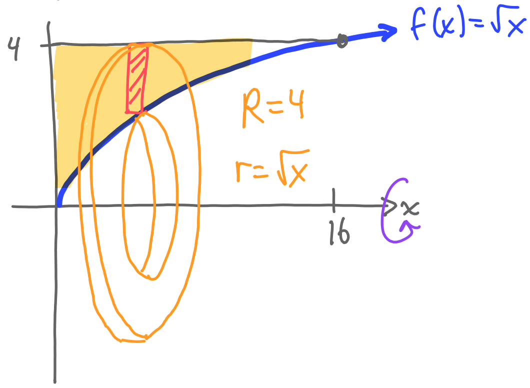
Volume:

$$V = \int_0^1 \pi\left((3-x^2)^2 - (1+x)^2\right)dx$$

$$= \boxed{\frac{73}{15}\pi}$$

6. Sketch the region bounded by the graph $f(x) = \sqrt{x}$, $y = 4$, and $x = 0$.

(a) If this region is rotated around the x-axis, what is the volume of the resulting object?



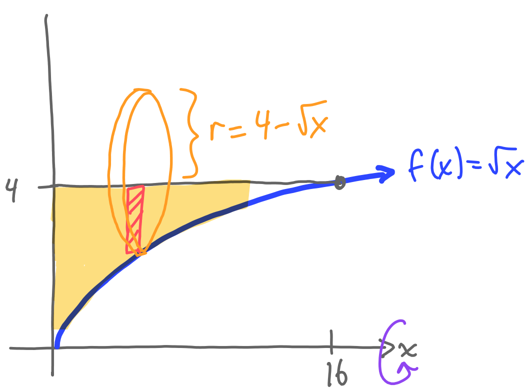
Volume element:

$$\Delta V = \pi\left(4^2 - (\sqrt{x})^2\right)\Delta x$$

$$\text{Volume} = \int_0^{16} \pi(16-x)\,dx$$

$$= \boxed{128\pi}$$

(b) If this region is rotated around the line $y = 4$, what is the volume of the resulting object?



Volume element:

$$\Delta V = \pi(4-\sqrt{x})^2\,\Delta x$$

Volume:

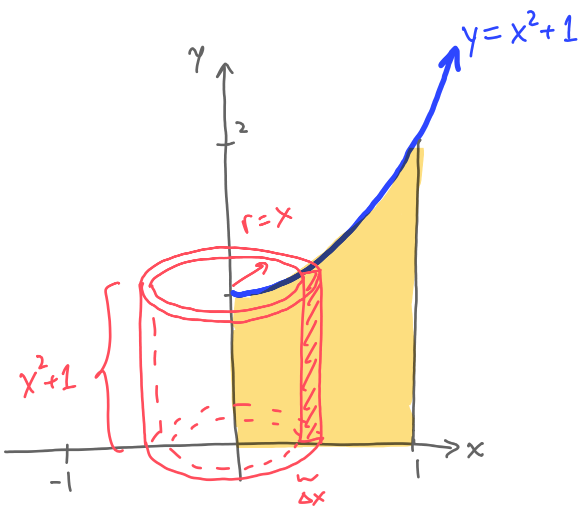
$$V = \int_0^{16} \pi(4-\sqrt{x})^2\,dx$$

$$= \boxed{\frac{128}{3}\pi}$$

7. Sketch the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$. If this region is rotated about the y-axis, why is it not convenient to compute the volume of the resulting solid using approximating disks or washers? What *else* could you use as your volume element in order to compute the volume of this solid? Try to find this volume!

👈 Recall Marissa's idea in #4.

We would have to compute the volume as a sum of two integrals.



Volume element: $\Delta V = 2\pi x\,(x^2+1)\,\Delta x$

$$\text{Volume} = \int_0^1 2\pi x(x^2+1)\,dx = \boxed{\frac{3}{2}\pi}$$