Exam 1 Review

1. *True or False?* If true, provide an argument that justifies your answer. If false, give a counterexample.

   A counterexample is a specific example that demonstrates that the statement fails.

   (a) A continuous function has a continuous derivative.

   (b) If $f(x) > 0$ and $f'(x) > 0$ for all x, then the right Riemann sum underestimates the integral $\displaystyle\int_a^b f(x)\ dx$.

   (c) $(a+b)^2 = a^2 + b^2$

   (d) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.

   (e) If f and g are continuous on $[a, b]$, then

   $$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

   (f) If f and g are continuous on $[a, b]$, then

   $$\int_a^b [f(x)g(x)]\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right).$$

2. Sketch the graph of a function $f(x)$ with the following properties:

   (a) $f(x)$ is discontinuous at $x = -2$

   (b) $f'(-1) = 0$

   (c) f is continuous at $x = 0$ but $f'(0)$ does not exist

   (d) $\displaystyle\int_0^2 f(x)\, dx = 0$

3. Show that each integral is equal to zero:

   (a) $\displaystyle\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\ d\theta$

   (b) $\displaystyle\int_0^{\sqrt{\pi}} t\cos(t^2)\ dt$

   (c) $\displaystyle\int_{-b}^{b} \frac{4x}{2+x^2}\ dx$ for any $b > 0$.

4. Let R be the region bounded by the graphs of $y = x + 2$ and $y = x^2$.

   (a) Sketch the region R.

   (b) Find the area of the region R.

   (c) Find the volume of the solid obtained by rotating R about the x-axis.

5. (a) Find the area under the graph of $f(t) = \dfrac{t}{1+t^2}$ between $t = 0$ and $t = x$. What happens to the area as x increases towards ∞?

   (b) Find the area under the graph of $f(t) = \dfrac{t}{(1+t^2)^2}$ between $t = 0$ and $t = x$. What happens to the area as x increases towards ∞?

6. Differentiate the following functions:

   (a) $f(x) = \left(3 + \sin^2(x)\right)^3$

   (b) $g(x) = \ln(\cos(2x))$

   (c) $h(x) = \displaystyle\int_{\sqrt{x}}^{1} (t^2 + 3t)\, dt$

7. Find each antiderivative, stating the technique that you use for each. If the technique is substitution, identify u. If the technique is integration by parts, identify u and dv.

   (a) $\displaystyle\int x \ln x \; dx$

   (b) $\displaystyle\int \frac{(\ln x)^2}{x} dx$

   (c) $\displaystyle\int xe^x \; dx$

   (d) $\displaystyle\int xe^{x^2+1} \; dx$

8. Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\, dt = 2\sin(x) - 1.$$