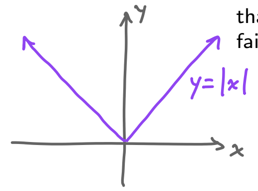
Exam 1 Review SOLUTIONS

1. *True or False?* If true, provide an argument that justifies your answer. If false, give a counterexample.

A counterexample is a specific example that demonstrates that the statement fails.

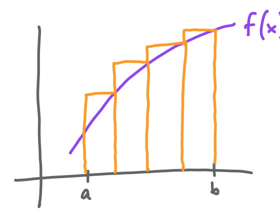
   (a) A continuous function has a continuous derivative.

   False: consider $y = |x|$, which is continuous but has a discontinuous derivative at $x = 0$.



   (b) If $f(x) > 0$ and $f'(x) > 0$ for all x, then the right Riemann sum underestimates the integral $\int_a^b f(x)\,dx$.

   False: If f is positive and increasing, then the right Riemann sum involves rectangles that are above the graph of f, producing an overestimate



WARNING! This is a common error!

   (c) $(a+b)^2 = a^2 + b^2$

   False! For example, if $a = b = 1$, then $(1+1)^2 = 4 \neq 2 = 1^2 + 1^2$.

   (d) If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

   True. If $f(x) \le g(x)$, then the signed area under the graph of $f(x)$ is less than the signed area under the graph of $g(x)$.

   (e) If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

   True. This is the additive property of integrals.

   (f) If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right).$$

   False! Consider $f(x) = x$ and $g(x) = x$. Then

$$\int_0^1 x \cdot x\,dx = \frac{1}{3} \neq \frac{1}{4} = \left(\int_0^1 x\,dx\right)\left(\int_0^1 x\,dx\right)$$

2. Sketch the graph of a function $f(x)$ with the following properties:

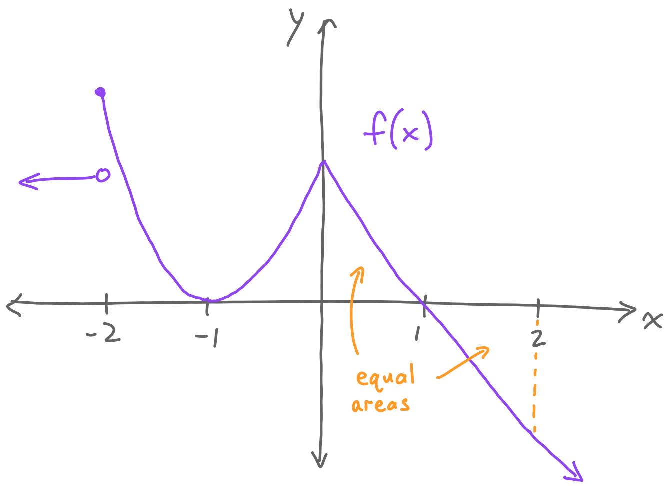
(a) $f(x)$ is discontinuous at $x = -2$

(b) $f'(-1) = 0$

(c) f is continuous at $x = 0$ but $f'(0)$ does not exist

(d) $\int_0^2 f(x)\,dx = 0$

Here is one such graph:



3. Show that each integral is equal to zero:

du

(a) $\frac{-1}{2}\int_0^{\pi} -2\cos^2(2\theta)\sin(2\theta)\,d\theta = \frac{-1}{2}\int_1^1 u^2\,du = -\frac{1}{2}\left[\frac{1}{3}u^3\right]_1^1 = 0$

$u = \cos(2\theta)$ If $\theta = 0$, then $u = 1$.

$du = -2\sin(2\theta)\,d\theta$ If $\theta = \pi$, then $u = 1$

du

(b) $\frac{1}{2}\int_0^{\sqrt{\pi}} 2t\cos(t^2)\,dt = \frac{1}{2}\int_0^{\pi}\cos(u)\,du = \frac{1}{2}\sin(u)\Big|_0^{\pi} = \frac{1}{2}(0-0) = 0$

$u = t^2$ If $x = 0$, then $u = 0$.

$du = 2t\,dt$ If $x = \sqrt{\pi}$, then $u = \pi$.

(c) $\int_{-b}^{b} \frac{4x}{2+x^2}\,dx$ for any $b > 0$.

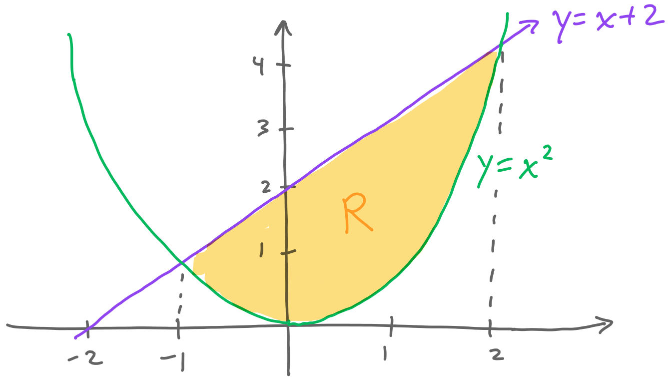
$u = 2 + x^2$

$du = 2x\,dx$

$\int_{-b}^{b}\frac{4x}{2+x^2}\,dx = \int_{2+b^2}^{2+b^2}\frac{2}{u}\,du = \Big[2\ln|u|\Big]_{2+b^2}^{2+b^2} = 0$

4. Let R be the region bounded by the graphs of $y = x + 2$ and $y = x^2$.

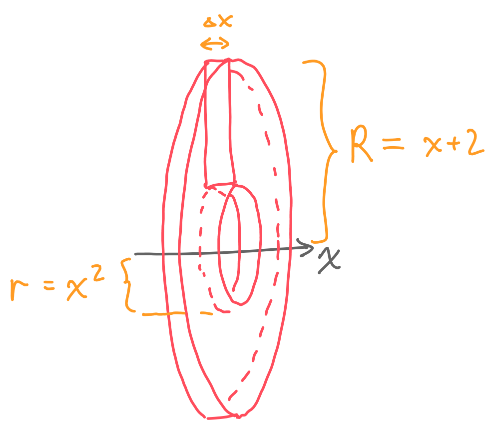
(a) Sketch the region R.



(b) Find the area of the region R.

$$\text{Area} = \int_{-1}^{2} \left(x + 2 - x^2\right) dx = \left[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\right]_{-1}^{2} = \frac{9}{2}$$

(c) Find the volume of the solid obtained by rotating R about the x-axis.



Volume element: $\Delta V = \pi\left((x+2)^2 - (x^2)^2\right)\Delta x$

Volume: $V = \displaystyle\int_{-1}^{2} \pi\left((x+2)^2 - (x^2)^2\right) dx$

$$V = \frac{124}{15}\pi$$

5. (a) Find the area under the graph of $f(t) = \dfrac{t}{1+t^2}$ between $t = 0$ and $t = x$. What happens to the area as x increases towards ∞?

$$\text{Area} = \int_0^x \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_1^{1+x^2} \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_1^{1+x^2} = \frac{1}{2}\Big(\ln(1+x^2) - \ln(1)\Big)$$

$u = 1+t^2$, $du = 2t\,dt$

$$\text{Area} = \frac{1}{2}\ln(1+x^2)$$

As x increases towards ∞, this area also increases towards ∞.

(b) Find the area under the graph of $f(t) = \dfrac{t}{(1+t^2)^2}$ between $t = 0$ and $t = x$. What happens to the area as x increases towards ∞?

$$\text{Area} = \int_0^x \frac{t}{(1+t^2)^2}\,dt = \frac{1}{2}\int_1^{1+x^2} \frac{1}{u^2}\,du = \frac{-1}{2u}\Big|_1^{1+x^2} = \frac{-1}{2(1+x^2)} - \frac{-1}{2}$$

$u = 1+t^2$, $du = 2t\,dt$

$$\text{Area} = \frac{1}{2} - \frac{1}{2(1+x^2)}$$

As x increases towards ∞, this area converges to $\frac{1}{2}$.

6. Differentiate the following functions:

(a) $f(x) = \big(3 + \sin^2(x)\big)^3$

$$f'(x) = 3\big(3 + \sin^2(x)\big)^2\big(2\sin(x)\cos(x)\big)$$

(b) $g(x) = \ln(\cos(2x))$

$$g'(x) = \frac{1}{\cos(2x)}\big(-2\sin(2x)\big) = -2\tan(2x)$$

(c) $h(x) = \displaystyle\int_{\sqrt{x}}^{1} (t^2 + 3t)\,dt$ $= -\displaystyle\int_1^{\sqrt{x}} (t^2+3t)\,dt$

$$h'(x) = -\Big((\sqrt{x})^2 + 3\sqrt{x}\Big)\left(\frac{1}{2\sqrt{x}}\right) = \frac{-x - 3\sqrt{x}}{2\sqrt{x}}$$

7. Find each antiderivative, stating the technique that you use for each. If the technique is substitution, identify u. If the technique is integration by parts, identify u and dv.

(a) $\int x \ln x \, dx = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x \, dx$

integrate by parts:

$u = \ln x \qquad v = \frac{1}{2}x^2$

$du = \frac{1}{x}dx \qquad dv = x \, dx$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

(b) $\int \frac{(\ln x)^2}{x} dx = \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$

Substitution:

$u = \ln x$

$du = \frac{1}{x}dx$

(c) $\int xe^x \, dx = xe^x - \int e^x dx = xe^x - e^x + C$

integrate by parts:

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

(d) $\int xe^{x^2+1} \, dx = \frac{1}{2}\int e^u \, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+1} + C$

substitution:

$u = x^2 + 1$

$du = 2x \, dx$

8. Find a function f and a value of the constant a such that

$$2\int_a^x f(t) \, dt = 2\sin(x) - 1.$$

Let F be an antiderivative of f, so $F' = f$.

Then $2\int_a^x f(t) \, dt = 2F(x) - 2F(a)$ by the FTC.

Suppose $F(x) = \sin(x)$. Then $f(x) = F'(x) = \cos(x)$

We also need $2F(a) = 2\sin(x) = 1$, so we could choose $a = \frac{\pi}{6}$.

Thus, a solution is $f(x) = \cos(x)$ and $a = \frac{\pi}{6}$.