MATH 126 A/B — 22 October 2025

POWER SERIES

Most important example:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{for } -1 < x < 1$$

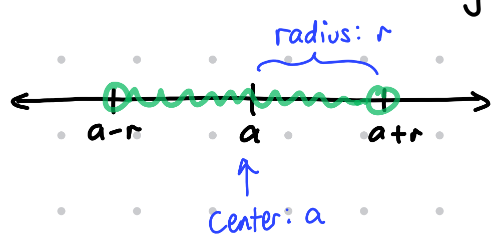
We can obtain other power series by replacing x with other expressions. For example, replace x with $-x^2$.

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \frac{1}{1+x^2} \quad \text{for } -1 < x < 1.$$

FACT: Power series can be "centered" at $x = a$:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

In this case, the interval of convergence is centered at $x = a$:



Conclusion from the long dialogue on the worksheet.

If there is a power series centered at $x = 0$ that converges to $f(x)$, then it has to be $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

($f^{(n)}(0)$ is the n^{th} derivative of $f(x)$ evaluated at $x = 0$)

Power Series

1. Do not forget that, for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \quad \text{converges to (i.e., } \textit{equals}\text{)} \quad \frac{1}{1-x}. = (1-x)^{-1}$$

(a) Can you figure out what this series converges to?

How is this related to the series above? Where have you seen nx^{n-1} before?

derivative!

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

$$= \frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}$$

(b) Can you figure out what this series converges to?

Where have you seen $\frac{x^n}{n}$ before?

antiderivative!

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \ldots$$

$$= \int \frac{1}{1-x}\,dx = -\ln|1-x| + C$$

To solve for C, set $x=0$. Then $0 = -\ln|1-0| + C$, so $0 = C$.

So: $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln|1-x|$

2. Cleo: Hey, look at this weird series:

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n = (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \frac{1}{4}(x-2)^4 + \frac{1}{5}(x-2)^5 + \frac{1}{6}(x-2)^6 + \ldots$$

Sundar: We have never seen one like this before. But it kind of reminds me of the series in #1(b).

Group Chat: Why do you think Sundar is reminded of $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

Starts with "S" and is 12 letters long.

SUBSTITUTION: replace x with $x-2$

Erez: The interval of x-values for which $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges is $\underline{-1 < x < 1}$.

replace x with $x-2$

$$-1 < x-2 < 1$$
$$+2 \qquad +2 \qquad +2$$
$$1 < x < 3$$

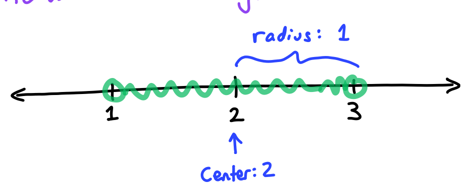
Group task: Assume Erez is correct. Then:

(a) Determine the interval of x-values for which $\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$ converges.

(b) What number is at the *center* of that interval?

(c) What is the radius of the interval of convergence?

Interval of Convergence:



3. Group Conjecture: If we used the ratio test to determine the interval of x-values for which a series that looks like $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges, the number at the *center* of that interval would be

$$x = \underline{\quad a \quad}.$$

ratio test:

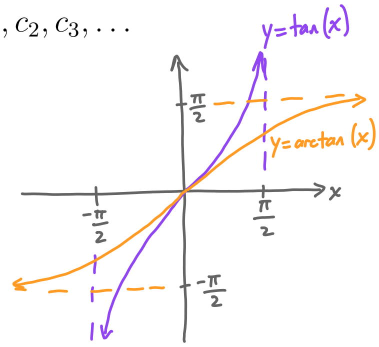
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}(x-a)\right| = \text{Const}\cdot|x-a| < 1 \quad \text{if} \quad \underbrace{|x-a|}_{\text{center of the interval of convergence}} < \underbrace{\frac{1}{\text{Const}}}_{\text{radius of the interval of convergence}}$$

4. Recall that $\boxed{\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}}$.

Find a power series that converges to $\arctan(x)$. That is, find coefficients $c_0, c_1, c_2, c_3, \ldots$ such that

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad = \quad \arctan(x)$$

for the values of x that allow the series to converge.



From previously: $1 - x^2 + x^4 - x^6 + x^8 - \cdots = \dfrac{1}{1+x^2}$ for $-1 < x < 1$

Antiderivative: $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots = \arctan(x) + C$ for $-1 < x < 1$

Set $x = 0$: $0 - \dfrac{0^3}{3} + \dfrac{0^5}{5} - \dfrac{0^7}{7} + \cdots = \arctan(0) + C$

$$0 = 0 + C$$

$$0 = C$$

Therefore,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 < x < 1$$

5. A conversation that will change your perspective on mathematics:

Ava: Can you believe we now have power series that converge to all of these functions?

$$\frac{1}{1-x} \qquad \frac{1}{1-x^2} \qquad \frac{1}{1+x^2} \qquad \frac{1}{(1-x)^2} \qquad \arctan(x)$$

Milo: That's REALLY cool. I wonder if you can pick *any* function $f(x)$ and do this.

Sam: So if $f(x) = \sin x$, are you proposing we should attempt to *find* a power series that converges to $\sin x$?

Milo: I am! Do you think there is a power series out there that converges to $f(x) = e^x$?

Ava: I think you're crazy. But, the good kind of crazy. I will do this experiment with you. So...you are saying that you want to TRY to get

$$f(x) \quad \textit{equals} \quad c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots ?$$

Milo: Yes, let's at least try! What is the worst that could happen?

Sam: Your idea could end up not working at all and then we would have wasted our time!

Ava: Let's at least try! Let's assume your crazy idea works, Milo. Then, if we try to let $x = 0$, we *have to* get

$$f(0) \quad \textit{equals} \quad c_0 + \underbrace{c_1 \cdot 0 + c_2 \cdot 0^2 + c_3 \cdot 0^3 + c_4 \cdot 0^4 + c_5 \cdot 0^5 + \cdots}_{\text{zero!}} .$$

Group Chat: Oooh! Ava just figured something out about one of the coefficients $c_0, c_1, c_2, c_3, \ldots$. What interesting fact did Ava uncover?

$$c_0 = f(0)$$

Milo: Well for this idea to work, at least we know that c_0 has to equal $f(0)$. That's progress!

Sam: But Milo, you *only* found out what c_0 has to equal! You still need to figure out what the values of c_1, c_2, c_3, etc. are equal to.

Milo: Oh yeah...and there are infinitely many of those. I am starting to lose hope.

Ava: OOOOH! Don't give up yet! Let's take the *derivative*, and *then* plug in $x = 0$:

$$\begin{aligned}
\text{Starting point:} \quad & f(x) \quad \textit{equals} \quad c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
\text{Take the derivative:} \quad & f'(x) \quad \textit{equals} \quad c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots \\
\text{Now let } x = 0\text{:} \quad & f'(0) \quad \textit{equals} \quad c_1 + \underbrace{c_2 \cdot 0 + c_3 \cdot 0^2 + c_4 \cdot 0^3 + c_5 \cdot 0^4 + \cdots}_{\text{zero!}}
\end{aligned}$$

Group Chat: What did Ava just discover about one of the coefficients $c_0, c_1, c_2, c_3, \ldots$?

$$c_1 = f'(0)$$

Milo: WOW! Now we know what c_1 would need to equal. Let's keep doing this procedure!

Group Chat: What do you think Milo means by "let's keep doing this procedure?"

Differentiate, then set $x = 0$.
Repeat!

Sam: OK! Now take the *second derivative*, and then plug in $x = 0$:

$$\begin{array}{rlll} \text{From before:} & f'(x) & \textit{equals} & c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots \\ \text{Differentiate again:} & f''(x) & \textit{equals} & 2c_2 + 2\cdot 3c_3x + 3\cdot 4c_4x^2 + 4\cdot 5c_5x^3 + \cdots \\ \text{Now let } x = 0\text{:} & f''(0) & \textit{equals} & 2c_2 + \underbrace{2\cdot 3c_3\cdot 0 + 3\cdot 4c_4\cdot 0^2 + 4\cdot 5c_5\cdot 0^3 + \cdots}_{\text{zero!}} \end{array}$$

Ava: Now we can solve for c_2.

Group Chat: What does c_2 need to equal to get this all to work?

$f''(0) = 2c_2$, so $c_2 = \frac{1}{2} f''(0)$

Group Task: Try using the *third* derivative with $x = 0$ to solve for c_3. What must c_3 be equal to in order to get this crazy idea to work?

$$f'''(x) \text{ equals } 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4x + 3\cdot 4\cdot 5c_5x^2 + \cdots$$

$$f'''(0) \text{ equals } 2\cdot 3c_3 + \underbrace{2\cdot 3\cdot 4c_4\cdot 0 + 3\cdot 4\cdot 5c_5\cdot 0^2 + \cdots}_{\text{zero!}}$$

So $f'''(0) = 6c_3$, which means $c_3 = \frac{1}{6} f'''(0)$

Group Discussion: Take a guess at what c_4 must equal. How can you generalize this procedure in order to solve for c_n? What must c_n be equal to in order to get this to work?

$$c_n = \frac{f^{(n)}(0)}{n!}$$

nth derivative of $f(x)$, evaluated at $x = 0$

6. Use what you learned above to write down the first several terms of the power series that converges to $f(x) = \sin(x)$.

We'll come back to this next time.