Power Series

1. Do not forget that, for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \quad \text{converges to (i.e., } \textit{equals}\text{)} \quad \frac{1}{1-x}.$$

(a) Can you figure out what this series converges to?

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

☞ How is this related to the series above? Where have you seen nx^{n-1} before?

(b) Can you figure out what this series converges to?

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \ldots$$

☞ Where have you seen $\frac{x^n}{n}$ before?

2. Cleo: Hey, look at this weird series:

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n = (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \frac{1}{4}(x-2)^4 + \frac{1}{5}(x-2)^5 + \frac{1}{6}(x-2)^6 + \ldots$$

Sundar: We have never seen one like this before. But it kind of reminds me of the series in #1(b).

Group Chat: Why do you think Sundar is reminded of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$?

☞ Starts with "S" and is 12 letters long.

Erez: The interval of x-values for which $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges is $-1 < x < 1$.

Group task: Assume Erez is correct. Then:

(a) Determine the interval of x-values for which $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$ converges.

(b) What number is at the *center* of that interval?

(c) What is the radius of the interval of convergence?

3. **Group Conjecture:** If we used the ratio test to determine the interval of x-values for which a series that looks like $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges, the number at the *center* of that interval would be

$$x = \underline{\hspace{5em}}.$$

4. Recall that $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$.

Find a power series that converges to $\arctan(x)$. That is, find coefficients $c_0, c_1, c_2, c_3, \ldots$ such that

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad = \quad \arctan(x)$$

for the values of x that allow the series to converge.

5. A conversation that will change your perspective on mathematics:

Ava: Can you believe we now have power series that converge to all of these functions?

$$\frac{1}{1-x} \qquad \frac{1}{1-x^2} \qquad \frac{1}{1+x^2} \qquad \frac{1}{(1-x)^2} \qquad \arctan(x)$$

Milo: That's REALLY cool. I wonder if you can pick *any* function $f(x)$ and do this.

Sam: So if $f(x) = \sin x$, are you proposing we should attempt to *find* a power series that converges to $\sin x$?

Milo: I am! Do you think there is a power series out there that converges to $f(x) = e^x$?

Ava: I think you're crazy. But, the good kind of crazy. I will do this experiment with you. So...you are saying that you want to TRY to get

$$f(x) \quad \textit{equals} \quad c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots ?$$

Milo: Yes, let's at least try! What is the worst that could happen?

Sam: Your idea could end up not working at all and then we would have wasted our time!

Ava: Let's at least try! Let's assume your crazy idea works, Milo. Then, if we try to let $x = 0$, we *have to* get

$$f(0) \quad \textit{equals} \quad c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + c_3 \cdot 0^3 + c_4 \cdot 0^4 + c_5 \cdot 0^5 + \cdots .$$

Group Chat: Oooh! Ava just figured something out about one of the coefficients $c_0, c_1, c_2, c_3, \ldots$. What interesting fact did Ava uncover?

Milo: Well for this idea to work, at least we know that c_0 has to equal $f(0)$. That's progress!

Sam: But Milo, you *only* found out what c_0 has to equal! You still need to figure out what the values of c_1, c_2, c_3, etc. are equal to.

Milo: Oh yeah...and there are infinitely many of those. I am starting to lose hope.

Ava: OOOOH! Don't give up yet! Let's take the *derivative*, and *then* plug in $x = 0$:

$$\begin{array}{rlll}
\text{Starting point:} & f(x) & \textit{equals} & c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots \\
\text{Take the derivative:} & f'(x) & \textit{equals} & c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots \\
\text{Now let } x = 0\text{:} & f'(0) & \textit{equals} & c_1 + c_2 \cdot 0 + c_3 \cdot 0^2 + c_4 \cdot 0^3 + c_5 \cdot 0^4 + \cdots
\end{array}$$

Group Chat: What did Ava just discover about one of the coefficients $c_0, c_1, c_2, c_3, \ldots$?

Milo: WOW! Now we know what c_1 would need to equal. Let's keep doing this procedure!

Group Chat: What do you think Milo means by "let's keep doing this procedure?"

Sam: OK! Now take the *second derivative*, and then plug in $x = 0$:

$$\begin{array}{rlll} \text{From before:} & f'(x) & \textit{equals} & c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots \\ \text{Differentiate again:} & f''(x) & \textit{equals} & 2c_2 + 2\cdot 3c_3x + 3\cdot 4c_4x^2 + 4\cdot 5c_5x^3 + \cdots \\ \text{Now let } x=0\text{:} & f''(0) & \textit{equals} & 2c_2 + 2\cdot 3c_3\cdot 0 + 3\cdot 4c_4\cdot 0^2 + 4\cdot 5c_5\cdot 0^3 + \cdots \end{array}$$

Ava: Now we can solve for c_2.

Group Chat: What does c_2 need to equal to get this all to work?

Group Task: Try using the *third* derivative with $x = 0$ to solve for c_3. What must c_3 be equal to in order to get this crazy idea to work?

Group Discussion: Take a guess at what c_4 must equal. How can you generalize this procedure in order to solve for c_n? What must c_n be equal to in order to get this to work?

6. Use what you learned above to write down the first several terms of the power series that converges to $f(x) = \sin(x)$.