MATH 126 A/B — 27 Oct. 2025

Exam 2 Practice Problems

1. Determine whether each of the following statements is *always true*, *sometimes true*, or *never true*. Explain your reasoning.

(a) If $f(x)$ is continuous on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ converges.

SOMETIMES TRUE

For example, $\int_1^\infty \frac{1}{x^2}\,dx$ converges but $\int_1^\infty \frac{1}{x}\,dx$ diverges.

(b) If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)\,dx = -f(0)$.

ALWAYS TRUE:

$$\int_0^\infty f'(x)\,dx = \lim_{b\to\infty} \int_0^b f'(x)\,dx = \lim_{b\to\infty} \Big[f(x)\Big]_0^b = \lim_{b\to\infty} \Big[f(b) - f(0)\Big] = \Big(\lim_{b\to\infty} f(b)\Big) - f(0) = -f(0)$$

(c) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^\infty a_n$ converges.

SOMETIMES TRUE

For example, $\sum_{n=1}^\infty \left(\frac{1}{2}\right)^n$ converges but $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

(d) If the series $\sum_{n=1}^\infty a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

ALWAYS TRUE

If $\lim_{n\to\infty} a_n \neq 0$, then the divergence test says that $\sum_{n=1}^\infty a_n$ must diverge.

So if $\sum_{n=1}^\infty a_n$ converges, it must be that $\lim_{n\to\infty} a_n = 0$.

2. Evaluate each integral or show that it is divergent:

(a) $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x}\,dx = \lim_{b\to\infty} \left[-xe^{-x}\Big|_0^b - \int_0^b -e^{-x}\,dx\right] = \lim_{b\to\infty} \Big[-xe^{-x} - e^{-x}\Big]_0^b$

improper since ∞ is a bound

integrate by parts: $u = x$, $du = dx$, $v = -e^{-x}$, $dv = e^{-x}dx$

$$= \lim_{b\to\infty} \Big[(-be^{-b} - e^{-b}) - (0 - e^{-0})\Big] = 0 + 1 = 1$$

(b) $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} = \lim_{a\to1^+} \int_a^2 \frac{dx}{x\cdot\ln(x)} = \lim_{a\to1^+} \int_{\ln(a)}^{\ln(2)} \frac{1}{u}\,du = \lim_{a\to1^+} \Big[\ln(u)\Big]_{\ln(a)}^{\ln(2)} = \lim_{a\to1^+} \Big[\ln(\ln(2)) - \ln(\ln(a))\Big] = \text{DNE}$

improper since $\frac{1}{x\ln(x)}$ has a vertical asymptote at $x=1$

let $u = \ln(x)$, so $du = \frac{1}{x}dx$

As $a \to 1^+$, $\ln(a) \to 0^+$, so $\ln(\ln(a)) \to -\infty$

The integral diverges.

3. Let $a_n = \dfrac{\ln(n)}{\sqrt{n}}$ for each positive integer n.

(a) Does the sequence $\{a_n\}$ converge or diverge? Explain.

Since $\sqrt{n}$ grows faster than $\ln(n)$,

the sequence $\left\{\frac{\ln(n)}{\sqrt{n}}\right\}$ converges to zero.

(b) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Explain.

Note that $\frac{\ln(n)}{\sqrt{n}} > \frac{1}{\sqrt{n}}$ for $n \geq 3$ Thus, $\sum_{n=3}^{\infty} \frac{\ln(n)}{\sqrt{n}} > \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$.

But $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $(p=\frac{1}{2})$ So $\sum_{n=3}^{\infty} \frac{\ln(n)}{\sqrt{n}}$ is larger than a divergent

series, so it must also diverge.

4. Determine if the following series converges or not. If it does, then determine the sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$ This is a geometric series with $r = \frac{\pi}{3} > 1$,

so it diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n}$ This is a geometric series with $a=4$ and $r=\frac{2}{3} < 1$,

so it converges to $\frac{a}{1-r} = \frac{4}{1-\frac{2}{3}} = \frac{4}{\frac{1}{3}} = 12.$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n\, n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!}$ This is the Maclaurin series for e^x

with $x = \frac{1}{2}$. Therefore,

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = e^{\frac{1}{2}} = \sqrt{e} \approx 1.6487$$

5. (a) Differentiate the Maclaurin series for $\sin(x)$. Explain how this shows you that $\frac{d}{dx}\sin(x) = \cos(x)$.

first: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

differentiate: $\frac{d}{dx}\sin(x) = 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - 7\frac{x^6}{7!} + \cdots$

$$= \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}_{\text{Maclaurin series for } \cos(x)} = \cos(x)$$

Thus, $\frac{d}{dx}(\sin(x)) = \cos(x)$.

(b) Differentiate the Maclaurin series for e^x. Explain how this shows you that e^x is its own derivative.

first: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

differentiate: $\frac{d}{dx}(e^x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = e^x$$

Thus, $\frac{d}{dx}(e^x) = e^x$.

6. For the following, find the Taylor polynomial of degree n centered at a.

(a) $\sin(x)$ for $a = \frac{\pi}{2}$ and $n = 4$

$f(x) = \sin(x)$ so $f(\frac{\pi}{2}) = 1$

$f'(x) = \cos(x)$ so $f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin(x)$ so $f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos(x)$ so $f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(x) = \sin(x)$ so $f^{(4)}(\frac{\pi}{2}) = 1$

The Taylor polynomial of degree 4 for $\sin(x)$ centered at $a = \frac{\pi}{2}$ is thus:

$$f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3 + \frac{f^{(4)}(\frac{\pi}{2})}{4!}(x-\tfrac{\pi}{2})^4$$

$$= 1 + 0 - \tfrac{1}{2}(x-\tfrac{\pi}{2})^2 + 0 + \tfrac{1}{4!}(x-\tfrac{\pi}{2})^4$$

$$= 1 - \tfrac{1}{2}(x-\tfrac{\pi}{2})^2 + \tfrac{1}{24}(x-\tfrac{\pi}{2})^4$$

(b) $\sqrt{1+x}$ for $a = 3$ and $n = 2$

$f(x) = (1+x)^{1/2}$ so $f(3) = 4^{1/2} = 2$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ so $f'(3) = \frac{1}{2}(4)^{-1/2} = \frac{1}{2\cdot 2} = \frac{1}{4}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$ so $f''(3) = \frac{-1}{4}(4)^{-3/2} = \frac{-1}{4\, 2^3} = \frac{-1}{32}$

The Taylor polynomial of degree 2 for $\sqrt{1+x}$ centered at $a=3$ is then:

$$f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2$$

$$= 2 + \tfrac{1}{4}(x-3) - \frac{1}{32\cdot 2}(x-3)^2$$

$$= 2 + \frac{x-3}{4} - \frac{(x-3)^2}{64}$$

7. Find the interval of convergence of the following series:

(a) $\sum_{n=0}^{\infty} \frac{n}{b^n}(x-a)^n$ where $b > 0$ Let $c_n = \frac{n}{b^n}(x-a)^n$.

Ratio test: $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)}{n\cdot b}\right| = \frac{|x-a|}{b}$

The series converges if $\frac{|x-a|}{b} < 1$, which means $|x-a| < b$, so the center of convergence is a, and the radius of convergence is b. That is, $a-b < x < a+b$.

ENDPOINTS: At $x = a-b$, the series is $\sum_{n=0}^{\infty} \frac{n(-b)^n}{b^n} = \sum_{n=0}^{\infty} n(-1)^n$, which diverges.

At $x = a+b$, the series is $\sum_{n=0}^{\infty} \frac{n(b^n)}{b^n} = \sum_{n=0}^{\infty} n$, which diverges.

$\boxed{\text{Therefore, the interval of convergence is } (a-b, a+b)}$

(b) $\sum_{n=0}^{\infty} n!(x-a)^n$

Let $c_n = n!(x-a)^n$.

Ratio test: $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-a)^{n+1}}{n!\,(x-a)^n}\right| = \lim_{n\to\infty}\left|(n+1)(x-a)\right|$

This limit diverges unless $x = a$, in which case the limit is 0

Therefore, this series converges only at $x=0$.

8. The limit $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$ is tricky to evaluate, but not if you know about Maclaurin series! Use the Maclaurin series for $\cos(x)$ to evaluate the limit.

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots\right)}{x^2} = \lim_{x\to 0}\frac{\frac{x^2}{2}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}{x^2}$$

All the other terms have x to a power larger than 2

$$= \lim_{x\to 0}\frac{x^2\left(\frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots\right)}{x^2} = \lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots\right) = \frac{1}{2}$$

9. Recall that $\arctan(1) = \frac{\pi}{4}$. Use the Maclaurin series for $\arctan(x)$ to produce a series that converges to π.

Maclaurin series: $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ (for $|x| \le 1$)

Let $x = 1$: $\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Therefore: $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$