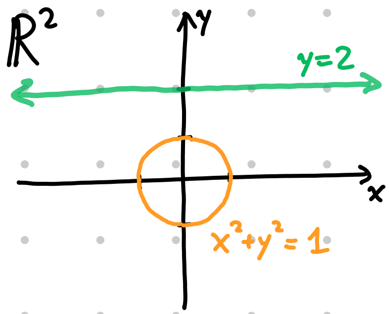
MATH 126 A/B — 31 October 2025

2D Coordinate System



Graphs consist of pairs of points (x,y).

What does it mean to graph $y=2$?

It means to plot all points (x,y) that satisfy $y=2$. So: $x=$ anything, $y=2$.

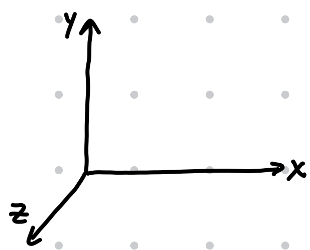
Another example: $x^2+y^2=1$

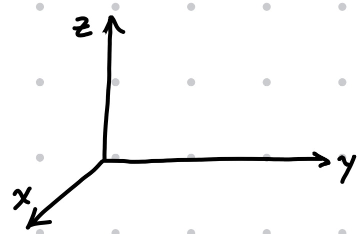
3D Coordinate System

DESMOS 3D plotter

Points: (x,y,z)

Axes can be drawn from different perspectives, but keep x,y oriented as in $\mathbb{R}^2$.





Distance Formula

The distance from $P=(x_1,y_1,z_1)$ to $Q=(x_2,y_2,z_2)$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Contour Maps and Level Curves

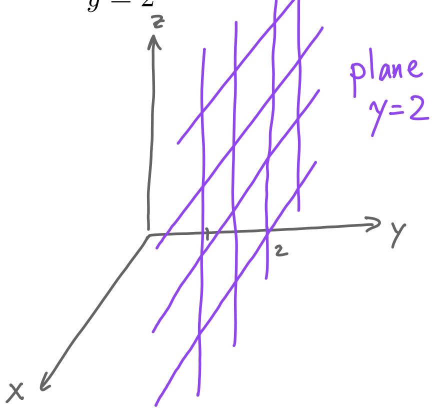
If $f(x,y)$ is the altitude at point (x,y), then each contour is $\underbrace{\text{the graph of } f(x,y)=k}$ for some constant k.

Such a graph is a level curve

Multivariable Functions

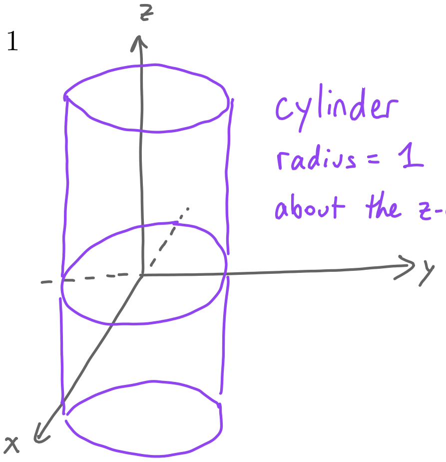
1. Sketch the 3D coordinate axes, and label each axis. Then draw the graphs of the following equations.

$y = 2$



plane
y=2

$x^2 + y^2 = 1$



cylinder
radius = 1
about the z-axis

2. Let $f(x, y) = \sqrt{x^2 - y^2}$. Note that this is a function of *two* variables, x and y!

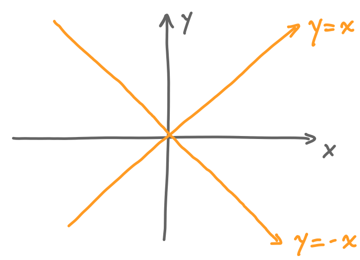
(a) What are all x and y values such that $f(x, y) = 0$? Draw the set of these values in the (x, y)-plane.

$0 = \sqrt{x^2 - y^2}$

$0 = x^2 - y^2$

$y^2 = x^2$

$y = \pm x$



Fix x = 1:

$z = \sqrt{1^2 - y^2}$

$z^2 = 1^2 - y^2$

$y^2 + z^2 = 1^2$

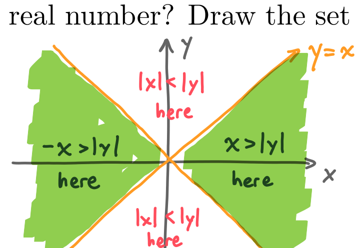
half circle
slice at x=1

"trace" of the
graph at x = 1

(b) What are all x and y values such that $f(x, y)$ is a real number? Draw the set of these values in the (x, y)-plane.

$f(x,y)$ is a real number if $x^2 > y^2$

that is, if $|x| > |y|$



(c) Choose some points (x, y) and compute $f(x, y)$. Where is $f(x, y)$ big? Where is $f(x, y)$ small?

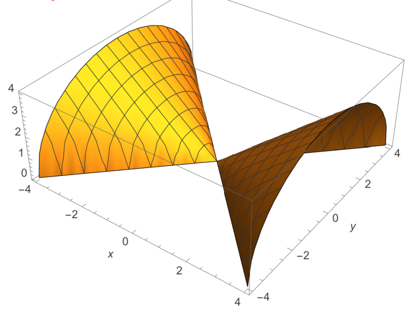
$f(x,y)$ is largest when
$|x|$ is big and
y is zero

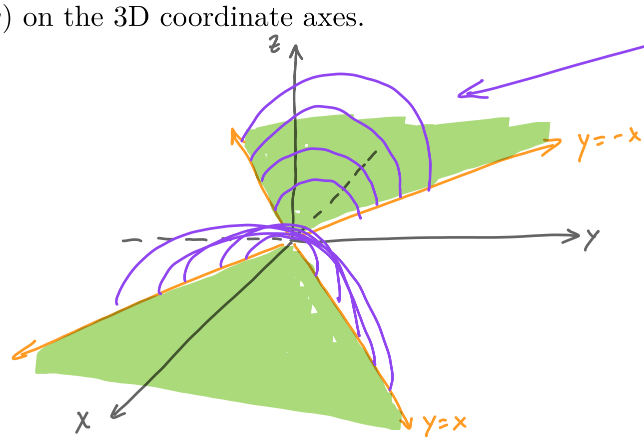
Also: Fix a value $x = c$. Then the graph $z = f(c, y)$ is $z = \sqrt{c^2 - y^2}$ which is $z^2 = c^2 - y^2$, which is $z^2 + y^2 = c^2$.
So cross sections (or "traces") of $f(x,y)$ perpendicular to the x-axis are half circles.

Everyone at your table can choose different points! Then compare answers.

(d) Sketch the graph of $f(x, y)$ on the 3D coordinate axes.

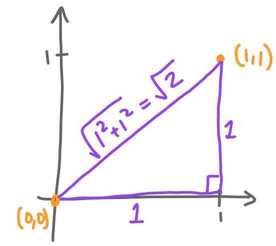
Graph from Mathematica:





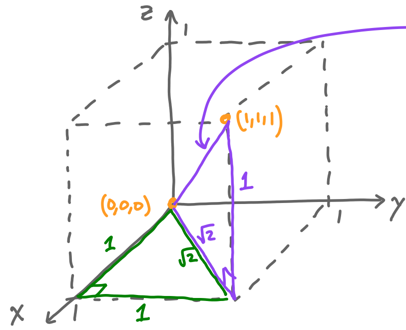
3. (a) What is the distance between $(0,0)$ and $(1,1)$ in 2D?

☝ Is there a right triangle hiding here?



(b) What is the distance between $(0,0,0)$ and $(1,1,1)$ in 3D?

☝ Where is the right triangle?



length: $\sqrt{\sqrt{2}^2 + 1^2}$

$= \sqrt{3}$

$= \sqrt{1^2+1^2+1^2}$

4. Consider the points $P = (1,-2,1)$, $Q = (5,1,1)$, and $R = (1,1,1)$.

(a) Find the distance between the points P and Q.

$$\text{distance} = \sqrt{(5-1)^2 + (1-(-2))^2 + (1-1)^2}$$

$$= \sqrt{4^2 + 3^2 + 0^2} = 5$$

(b) Find an equation of the sphere with radius 3 centered at P. $= (1,-2,1)$

↳ The set of points (x,y,z) whose distance from P is 3

Distance from (x,y,z) to P: $\sqrt{(x-1)^2 + (y-(-2))^2 + (z-1)^2} = 3$

or: $(x-1)^2 + (y+2)^2 + (z-1)^2 = 9$

(c) Does R lie within the sphere of radius 2 centered at P?

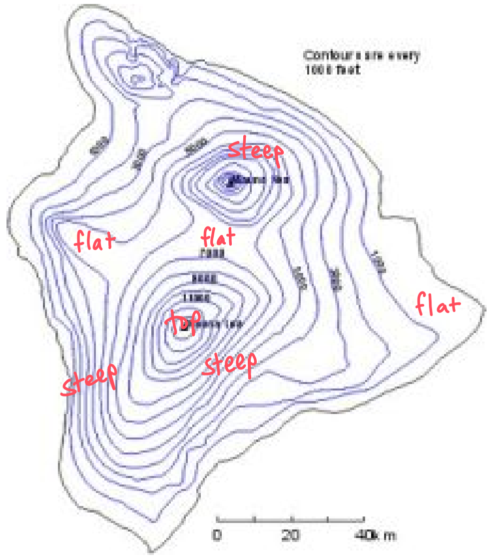
What is the distance from R to P?

$$\text{dist} = \sqrt{(1-1)^2 + (1-(-2))^2 + (1-1)^2} = \sqrt{0^2+3^2+0^2} = 3$$

Since the distance from R to P is 3,

R is <u>not</u> inside the circle of radius 2 centered at P.

5. The image at below is a contour map of the big island of Hawaii. It shows level curves on the surface of the island.



(a) Where is a (mostly) flat spot on the island?

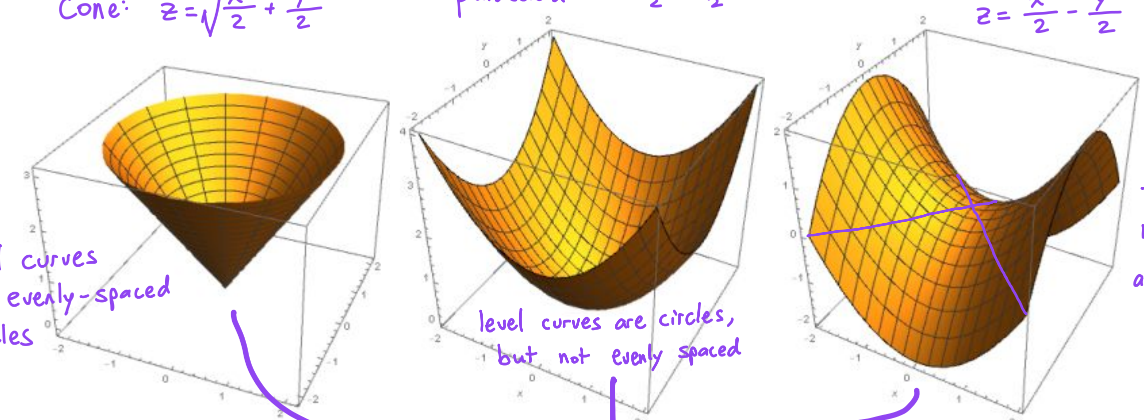
(b) Where are some steep places on the island?

6. Match the following 3D graphs with their level curves.

Cone: $z=\sqrt{\frac{x^2}{2}+\frac{y^2}{2}}$

paraboloid: $z=\frac{x^2}{2}+\frac{y^2}{2}$

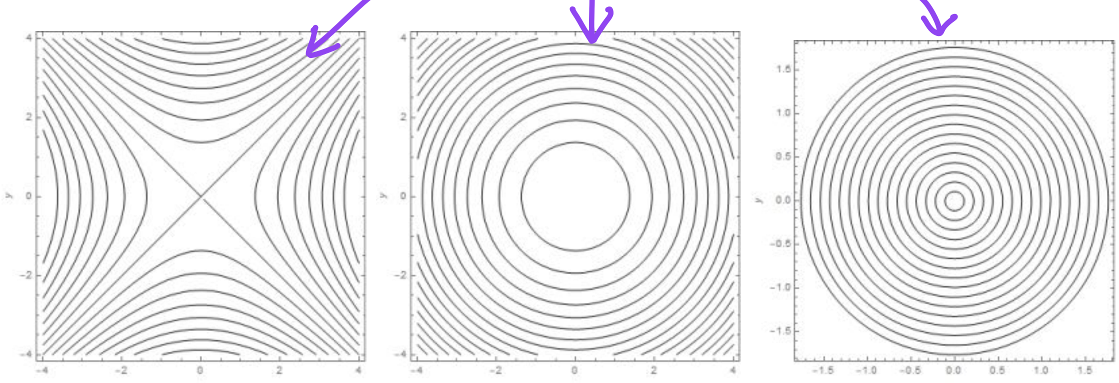
hyperbolic paraboloid: $z=\frac{x^2}{2}-\frac{y^2}{2}$



level curves are evenly-spaced circles

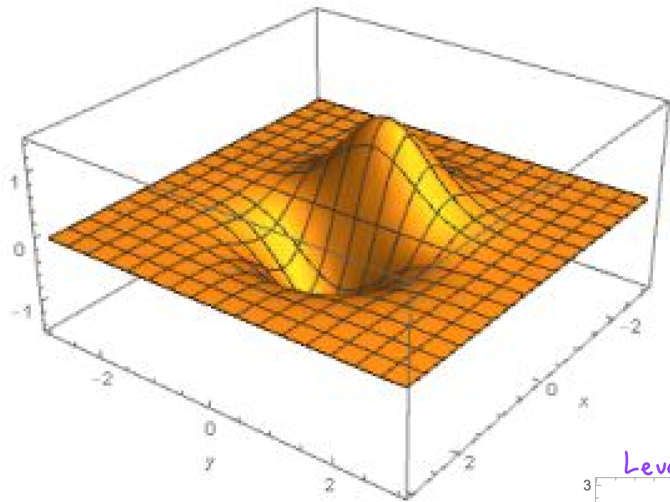
level curves are circles, but not evenly spaced

two lines through the middle point are level, and all other level sets are curves

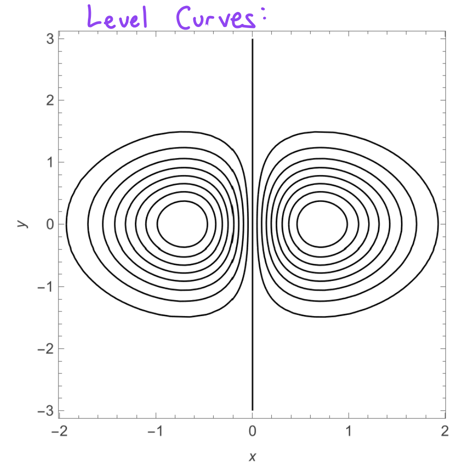


We didn't get to this in class.

7. The graph of $f(x,y) = -3xe^{-x^2-y^2}$ is pictured below.



Level Curves:



(a) Sketch the level curves of f.

The y-axis is a level curve.

The other level curves encircle either the hill or the depression.

(b) Suppose you fix $x = 0$. What does the cross-section of the graph look like for $x = 0$? This is called the $x = 0$ *trace* of the graph.

$$f(0,y) = -3(0)e^{-0^2-y^2} = 0$$

So the trace of $f(x,y)$ at $x=0$ is $z=0$.

The set of points $(0,y,0)$ is the line that is the y-axis.

(c) Now fix $y = 0$. What does the $y = 0$ trace look like?

$$f(x,0) = -3xe^{-x^2-0^2} = -3xe^{-x^2}$$

The trace of $f(x,y)$ at $y=0$ is $z=-3xe^{-x^2}$,

which looks like this: