MATH 126 A/B — 14 Nov. 2025

FIRST PARTIAL DERIVATIVES

If $f(x,y)$ is a function of two variables, then $f_x = \frac{\partial f}{\partial x}$ gives the slope of tangent lines to the graph of f in the x-direction, and $f_y = \frac{\partial f}{\partial y}$ gives the slope of tangent lines to the graph of f in the y-direction.

SECOND PARTIAL DERIVATIVES

If $f(x,y)$ is a function of two variables, there are four ways to take second derivatives of f:

x, then x: $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ ← concavity in the x-direction

y, then y: $f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ ← concavity in the y-direction

x, then y: $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

y, then x: $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

↑ left-to-right order

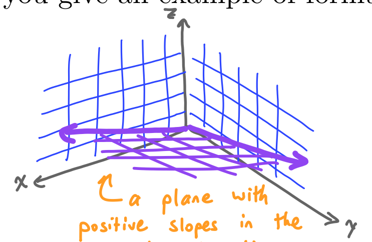
↑ right-to-left order

f_{xy} and f_{yx} are the mixed partial derivatives.

They indicate how the slope in the x- or y-direction changes as you move in the other direction.

Second Partial Derivatives

1. Sketch the graph of a function $z = f(x, y)$ whose derivatives f_x and f_y are always positive. Can you give an example of formula for such a function?



a plane with positive slopes in the x- and y-directions

Examples:

$z = x + y$

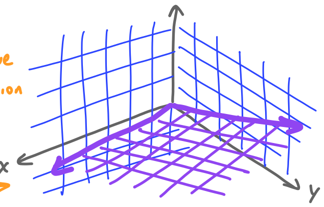
$z = \frac{x+y}{4}$

$z = e^x + e^y$

$z = 2^{x+y}$

2. Sketch the graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always positive. Can you give an example of formula for such a function?

A plane with negative slope in the x-direction and positive slope in the y-direction.



Examples:

$z = -x + y$

$z = \frac{y-x}{4}$

$z = -e^x + e^y$

We skipped this in class. →

3. Let $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$. Find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously. How would you describe the graph of $f(x, y)$ at these points?

$f_x = 2x + 4y - 4$ and $f_y = 4x + 2y + 16$

$f_x = 0$ means $2x + 4y - 4 = 0$

$\underline{x = 2 - 2y}$

$f_y = 0$ means $4x + 2y + 16 = 0$

$\underline{y = -2x - 8}$

substitute

$x = 2 - 2(-2x - 8)$

$x = 2 + 4x + 16$

$-3x = 18$ so $\underline{x = -6}$. Then $y = -2(-6) - 8$, so $\underline{y = 4}$.

$x = -6$ and $y = 4$

At $(-6, 4)$, the graph of f has tangent lines with slope zero in <u>both</u> the x- and y-directions.

4. **Chloe:** In Calculus I, we learned about more than just the first derivative $f'(x)$. We also learned about the second derivative $f''(x)$.

Bastian: I bet there are second derivatives in Calculus II, too! If you start with a function $f(x, y)$, you can take a partial derivative and then take another partial derivative!

Group chat: If you start with a function $f(x, y)$, how many *different ways* can you take a partial derivative and then take another partial derivative?

☜ There is more than one way!

Four: f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

Chloe: OK, so if you take the partial derivative of $f(x, y)$ with respect to x, and then take the partial derivative again with respect to x, what does that tell you about f?

Group chat: How would you answer Chloe's question?

The second partial derivative f_{xx} tells you the <u>concavity</u> of the graph of f in the x-direction.

Bastian: That makes sense. But what do the *other* second partial derivatives tell you about f?

Group chat: How would you answer Bastian's question?

f_{yy} tells you the concavity in the y-direction.

f_{xy} tells you how the slope in the x-direction changes when you move in the y-direction.

f_{yx} tells you how the slope in the y-direction changes when you move in the x-direction.

5. Find *all* second partial derivatives of $f(x,y) = 3xy^2 - 2y + 5x^2y^2$. Does anything surprise you about these partial derivatives?

$f_x = 3y^2 + 10xy^2$ $\qquad$ $f_y = 6xy - 2 + 10x^2y$

$f_{xx} = 10y^2$ $\qquad$ $f_{yy} = 6x + 10x^2$

$$f_{xy} = 6y + 20xy \xleftrightarrow{\text{SAME!}} f_{yx} = 6y + 20xy$$

class ended here

6. Find *all* second partial derivatives of $g(x,y) = \sqrt{x^2+y^2}$. Does anything surprise you about these partial derivatives?

$g_x = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = x(x^2+y^2)^{-1/2}$ $\qquad$ $g_y = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = y(x^2+y^2)^{-1/2}$

$$g_{xx} = 1(x^2+y^2)^{-1/2} + x\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}(2x) = \frac{(x^2+y^2) - x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$g_{yy} = 1(x^2+y^2)^{-1/2} + y\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}(2y) = \frac{(x^2+y^2) - y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$g_{xy} = x\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}(2y) = \frac{xy}{(x^2+y^2)^{3/2}} \xleftrightarrow{\text{SAME!}} g_{yx} = y\left(\tfrac{-1}{2}\right)(x^2+y^2)^{-3/2}(2y) = \frac{y^2}{(x^2+y^2)^{3/2}}$$

7. Find *all* second partial derivatives of $h(x,y) = e^x \tan(y)$. Does anything surprise you about these partial derivatives?

$h_x = e^x \tan(y)$ $\qquad$ $h_y = e^x \sec^2(y)$

$h_{xx} = e^x \tan(y)$ $\qquad$ $h_{yy} = e^x(2\sec(y)\cdot\sec(y)\tan(y)) = 2e^x\sec^2(y)\tan(y)$

$$h_{xy} = e^x \sec^2(y) \xleftrightarrow{\text{SAME!}} h_{yx} = e^x \sec^2(y)$$

8. How many second partial derivatives are there for the function $f(x,y,z) = ye^x + x\ln(z)$? Which ones do you think will be equal? Then compute these partial derivatives to check your hypothesis!

First partial derivatives:

$f_x = ye^x + \ln(z)$ $\qquad$ $f_y = e^x$ $\qquad$ $f_z = x \cdot \frac{1}{z} = \frac{x}{z}$

Second partial derivatives:

$f_{xx} = ye^x$ $\qquad$ $f_{yx} = e^x$ $\qquad$ $f_{zx} = \frac{1}{z}$

SAME!

$f_{xy} = e^x$ $\qquad$ $f_{yy} = 0$ $\qquad$ $f_{zy} = 0$

SAME! $\qquad$ SAME!

$f_{xz} = \frac{1}{z}$ $\qquad$ $f_{yz} = 0$ $\qquad$ $f_{zz} = \frac{-x}{z^2}$

NOTE: $f_{yz} = f_{zy}$ by Clairaut's theorem. It's a coincidence that f_{yy} is also equal to f_{yz} and f_{zy} here.