MATH 126 A/B — 3 December 2025

FUBINI'S THEOREM:

If $f(x,y)$ is a continuous function on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

You can integrate x first, then y.

Or you can integrate y first, then x.

7. Sketch the solid under the graph of $f(x,y) = \sqrt{4-x^2}$ and above the region $R = [-2,2] \times [0,3]$. Then evaluate the following integral.

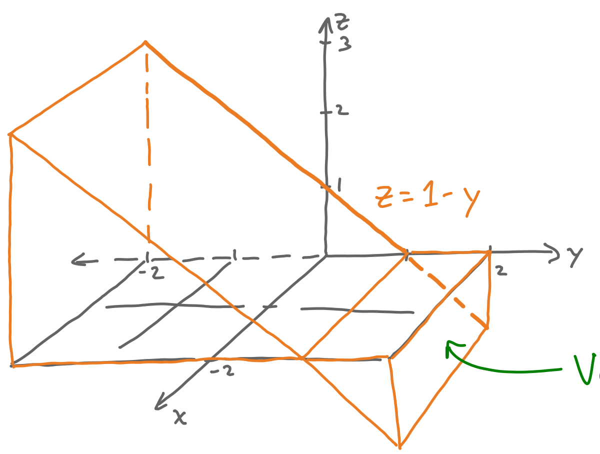
$$\iint\limits_R \sqrt{4-x^2}\, dA$$

(see Monday's notes for this solution)

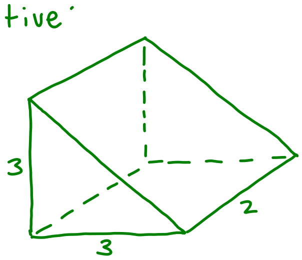
8. Sketch the solid between the graph of $f(x,y) = 1 - y$ and the region $R = [0,2] \times [-2,2]$. Then evaluate the following integral.

👈 Why does this problem say "between" instead of "under"?

$$\iint\limits_R (1-y)\, dA$$

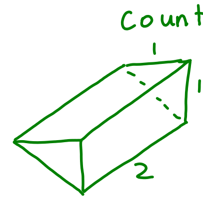


Volume above the xy-plane counts positive.



Volume $= \frac{1}{2}(3)(3)(2) = 9$

Volume below the xy-plane counts negative.



Volume $= \frac{1}{2}(1)(1)(2) = 2$

Therefore, $\displaystyle\iint\limits_R (1-y)\, dA = 9 - 1 = 8.$

We don't know a formula for the volume under $f(x,y) = x^2+y^2$

Iterated Integrals

1. Goal: We want to evaluate $\iint_R (x^2+y^2)\,dA$, which equals the volume underneath the graph of $f(x,y) = x^2+y^2$ and above the rectangle $R = [-2,2]\times[-2,2]$.

☞ R is the rectangle of points (x,y) where $-2 \le x \le 2$ and $-2 \le y \le 2$.

(a) Suppose that c is simply a constant and $f(x) = x^2 + c^2$. Write down and evaluate the integral that gives the area between $f(x)$ and the x-axis for $-2 \le x \le 2$.

$$\text{Area} = \int_{-2}^{2} (x^2+c^2)\,dx = \left[\tfrac{1}{3}x^3 + c^2 x\right]_{-2}^{2} = \left(\tfrac{1}{3}(2)^3 + c^2(2)\right) - \left(\tfrac{1}{3}(-2)^3 + c^2(-2)\right)$$

$$= \frac{8}{3} + 2c^2 + \frac{8}{3} + 2c^2 = \frac{16}{3} + 4c^2$$

(b) **Ava:** OK, this means that $\int_{-2}^{2}(x^2+y^2)\,dx = \dfrac{16}{3} + 4y^2$.

Milo: What, you computed a definite integral. Isn't the answer supposed to be a *number*?

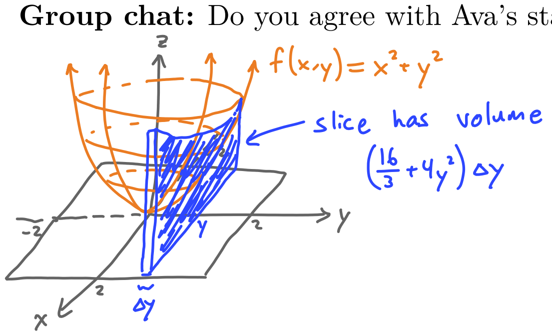
Group chat: How would you answer Milo's question?

When you integrate with respect to x, your answer can't have an x, but it can have a y in it.

(c) **Ava:** That must mean that the actual volume underneath $f(x,y) = x^2+y^2$ and above the rectangle R is given by

$$\int_{-2}^{2}\left(\frac{16}{3} + 4y^2\right)\,dy.$$

Group chat: Do you agree with Ava's statement? Explain.



Slice the volume into lots of thin slices, perpendicular to the y-axis, of thickness Δy. Each slice has volume $\left(\frac{16}{3}+4y^2\right)\Delta y$. Add up the volumes of the slices and let $\Delta y \to 0$, and the total volume is $\int_{-2}^{2}\left(\frac{16}{3}+4y^2\right)dy$.

(d) How can you evaluate $\iint_R (x^2+y^2)\,dA$?

$$\iint_R (x^2+y^2)\,dA = \int_{-2}^{2}\int_{-2}^{2}(x^2+y^2)\,dx\,dy = \int_{-2}^{2}\int_{-2}^{2}(x^2+y^2)\,dy\,dx$$

2. Now we want to compute $\iint_R \frac{xy^3}{x^2+1}\,dA$, where R is the rectangle $[0,1]\times[-3,3]$.

Let's play a fun game: split your group in half (or as close as you can get).

(a) One half of your group should compute the double integral this way: $\int_{-3}^{3}\int_0^1 \frac{xy^3}{x^2+1}\,dx\,dy$

inner integral: $\int_0^1 \frac{xy^3}{x^2+1}\,dx = \int_{u=1}^{u=2} \frac{1}{2}\cdot\frac{y^3}{u}\,du = \frac{y^3}{2}\ln(u)\Big|_{u=1}^{u=2} = \frac{y^3}{2}\left(\ln(2)-\ln(1)\right) = \frac{y^3\ln(2)}{2}$

$u = x^2+1, \quad du = 2x\,dx$

outer integral: $\int_{-3}^{3}\int_0^1 \frac{xy^3}{x^2+1}\,dx\,dy = \int_{-3}^{3}\frac{y^3\ln(2)}{2}\,dy = \frac{\ln(2)}{8}y^4\Big|_{y=-3}^{y=3} = \frac{\ln(2)}{8}\left(3^4-(-3)^4\right) = 0$

(b) The other half of your group should compute: $\int_0^1\int_{-3}^{3} \frac{xy^3}{x^2+1}\,dy\,dx$

Do you notice the difference?

inner integral: $\int_{-3}^{3}\frac{xy^3}{x^2+1}\,dy = \frac{x}{4(x^2+1)}y^4\Big|_{y=-3}^{y=3} = \frac{x}{4(x^2+1)}\left(3^4-(-3)^4\right) = 0$

outer integral: $\int_0^1\int_{-3}^{3}\frac{xy^3}{x^2+1}\,dy\,dx = \int_0^1 0\,dx = 0$

(c) Which half of your group had more fun?

I guess it depends on your definition of "fun."

I think part (b) was easier.

3. Evaluate $\iint_R x\cos(xy)\,dA$, where $R = \{(x,y) \mid 0\le x\le \pi, 0\le y\le 1\}$.

Does the order of integration matter?

If we integrate with respect to x first, we will have to do integration by parts. So let's try integrating with respect to y first.

$$\iint_R x\cdot\cos(xy)\,dA = \int_0^{\pi}\underbrace{\int_0^1 x\cdot\cos(xy)\,dy}_{\text{inner integral}}\,dx = \int_0^{\pi}\sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi)-(-\cos(0)) = -(-1)-(-1) = 2$$

inner integral

$\int_0^1 x\cdot\cos(xy)\,dy = \sin(xy)\Big|_{y=0}^{y=1} = \sin(x)-\sin(0) = \sin(x)$