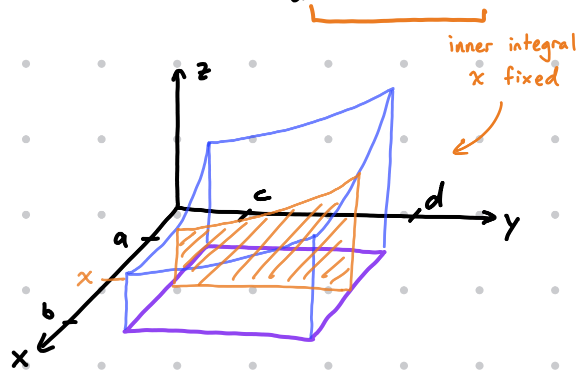
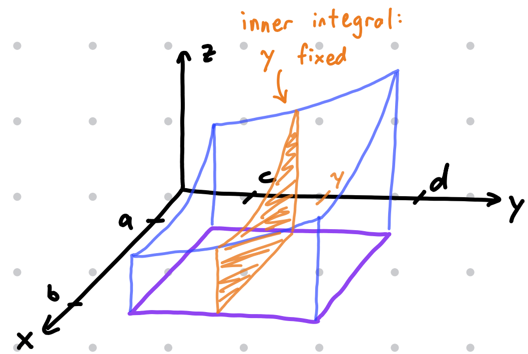
MATH 126 A/B — 5 December 2025

Double Integrals

Recall: double integral $\iint_R f(x,y)\,dA$ where $\underset{\uparrow \text{ rectangle}}{R} = [a,b]\times[c,d]$

evaluated as $\int_a^b \underbrace{\int_c^d f(x,y)\,dy}\,dx$ or $\int_c^d \underbrace{\int_a^b f(x,y)\,dx}\,dy$ ← iterated integrals



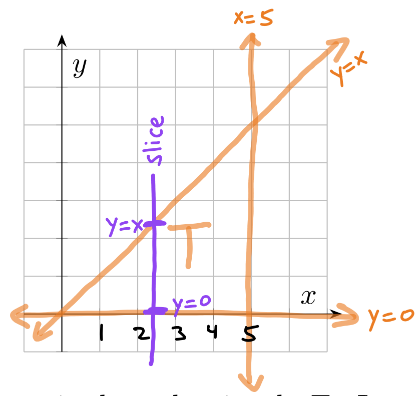


What if we want to integrate over a region that is <u>not</u> a rectangle?

Double Integrals over General Regions

1. Our goal is to find the volume underneath $f(x,y) = x^2 + y^2$ and above the triangular region T formed by the x-axis, the line $x = 5$, and the line $y = x$.

(a) Sketch the region T:



(b) Draw a slice parallel to the y-axis through triangle T. Let x be the value at which your slice crosses the x-axis. What are the minimum and maximum values of y on your slice of triangle T?

Your answer should have an x in it.

$$\underline{\quad 0 \quad} \le y \le \underline{\quad x \quad}$$

(c) Write an integral that gives the area below the graph of $f(x,y) = x^2 + y^2$ and above the slice you drew in part (b).

$$\int_0^x (x^2+y^2)\,dy = \left[x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=x} = \left(x^2(x) + \frac{1}{3}(x)^3\right) - \left(x^2(0) + \frac{1}{3}(0)^3\right) = \frac{4}{3}x^3$$

(d) What are the minimum and maximum values of x at which you could have drawn your slice in part (b)?

$$\underline{\quad 0 \quad} \le x \le \underline{\quad 5 \quad}$$

(e) Now integrate your result from part (c) over the range of x-values you found in part (d). How can you interpret this new integral?

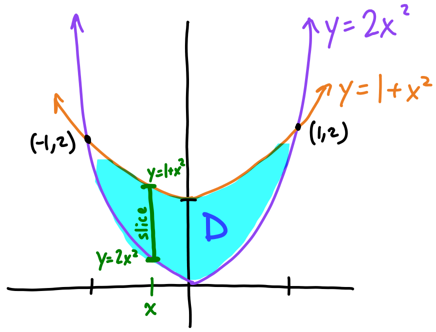
$$\int_0^5 \left(\int_0^x (x^2+y^2)\,dy\right) dx = \int_0^5 \frac{4}{3}x^3\,dx = \frac{1}{3}x^4 \Big|_{x=0}^{x=5} = \frac{5^4}{3} - \frac{0^4}{3} = \frac{625}{3}$$

(f) Use what you found above to rewrite the following double integral as an *iterated* integral with actual bounds of integration.

$$\iint_T (x^2+y^2)\,dA =$$

2. Evaluate the double integral $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$, by following the steps below.

(a) Sketch the region D.



(b) In order to evaluate the integral, we must first write it as an iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} (x+2y)\, dy\, dx$$

Determine the x values for when the two parabolas intersect. These values are a and b.

$2x^2 = 1 + x^2$ means $x^2 = 1$, so $x = \pm 1$.

Then: $a = -1$ and $b = 1$

(c) What are the inner bounds of integration $g_1(x)$ and $g_2(x)$? You can now write the double integral over D as an iterated integral.

Upper bound: $g_2(x) = 1 + x^2$

lower bound: $g_1(x) = 2x^2$

(d) Evaluate the iterated integral by integrating the inside integral first and then the outside integral last.

$$\iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx = \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx$$

$$= \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x + x\right]_{x=-1}^{x=1} = \frac{32}{15}$$

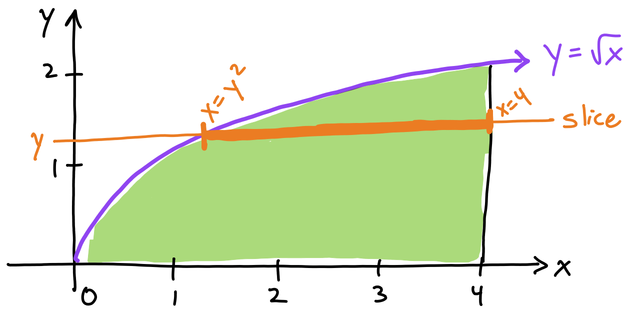
Inner integral: $\int_{2x^2}^{1+x^2} (x + 2y)\, dy = \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} = \left(x(1+x^2) + (1+x^2)^2\right) - \left(x(2x^2) + (2x^2)^2\right)$

$$= (x + x^3) + (1 + 2x^2 + x^4) - (2x^3 + 4x^4) = -3x^4 - x^3 + 2x^2 + x + 1$$

3. Sketch the region whose area is given by the iterated integral

$$\int_0^4 \int_0^{\sqrt{x}} 1 \, dy \, dx.$$

Then find another iterated integral whose value is equal to this integral, but with the order of integration $dx\,dy$. Show that both integrals yield the same value.

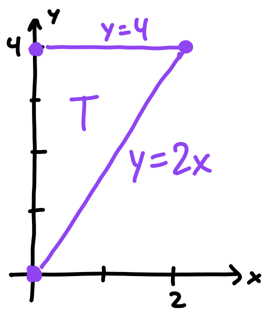


Integrating $dx\,dy$:

$$\int_0^2 \int_{y^2}^4 1 \, dx \, dy$$

4. Let T be the triangle with vertices $(0,0)$, $(0,4)$, and $(2,4)$. Evaluate the double integral:

$$\iint_T (x+y)\, dA$$



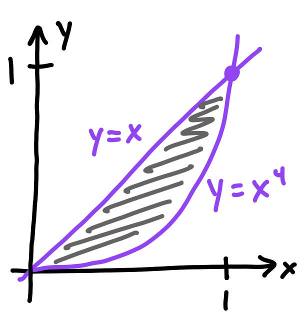
$$\iint_T (x+y)\, dA = \int_0^2 \int_{2x}^4 (x+y)\, dy\, dx = \int_0^2 (-4x^2+4x+8)\, dx$$

$$= \left[-\frac{4}{3}x^3 + 2x^2 + 8x \right]_0^2 = -\frac{4}{3}(2)^3 + 2(2)^2 + 8(2) = \frac{40}{3}$$

inner integral

$$\int_{2x}^4 (x+y)\, dy = \left[xy + \frac{1}{2}y^2 \right]_{y=2x}^{y=4} = \left(4x + \frac{4^2}{2}\right) - \left(x(2x) + \frac{(2x)^2}{2}\right) = 4x + 8 - (2x^2 + 2x^2) = -4x^2 + 4x + 8$$

5. Find the volume of the solid under the plane $z = x + 2y$ and above the region bounded by $y = x$ and $y = x^4$.



$$\text{Volume} = \int_0^1 \int_{x^4}^x (x+2y)\, dy\, dx = \int_0^1 (-x^8 - x^5 + 2x^2)\, dx$$

$$= \left[-\frac{1}{9}x^9 - \frac{1}{6}x^6 + \frac{2}{3}x^3 \right]_0^1 = -\frac{1}{9} - \frac{1}{6} + \frac{2}{3}$$

$$= \frac{7}{18} = 0.3\overline{8}$$

inner integral

$$\int_{x^4}^x (x+2y)\, dy = \left[xy + y^2 \right]_{y=x^4}^{y=x} = (x^2 + x^2) - (x^5 + x^8) = -x^8 - x^5 + 2x^2$$