3. The wave heights \( h \) in the open sea depend on the speed \( v \) of the wind and the length of time \( t \) that wind has been blowing at that speed (knots). Values of the function \( h = f(v, t) \) are recorded in feet in the following table.

<table>
<thead>
<tr>
<th>Wind speed</th>
<th>Duration (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>5 7 8 8 9 9 9</td>
</tr>
<tr>
<td>30</td>
<td>9 13 16 17 18 19 19</td>
</tr>
<tr>
<td>40</td>
<td>14 21 25 28 31 33 33</td>
</tr>
<tr>
<td>50</td>
<td>19 29 36 40 45 48 50</td>
</tr>
<tr>
<td>60</td>
<td>24 37 47 54 62 67 69</td>
</tr>
</tbody>
</table>

(a) Use the table to find a linear approximation to the wave height function when \( v \) is near 40 knots and \( t \) is near 20 hours.

\[
\begin{align*}
\text{linear approximation:} & \quad h(v, t) \approx 1.15(v-40) + 0.45(t-20) + 28 \\
\text{then} & \quad h(40, 20) \approx 1.15(40-40) + 0.45(20-20) + 28 = 33.25
\end{align*}
\]

(b) Using the linear approximation you just found, estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

\[
\begin{align*}
\text{estimated wave height} & \quad h(43, 24) \approx 1.15(43-40) + 0.45(24-20) + 28 \\
& \quad \approx 1.15(3) + 0.45(4) + 28 = 33.25
\end{align*}
\]

4. Let \( f(x, y) = \sqrt{y + \cos^2(x)} \). Use a linearization to approximate \( f(0.2, 0.1) \).

\[
\begin{align*}
\frac{df}{dx}(x,y) & = \frac{1}{2}(y + \cos^2(x))^{-1/2}(2\sin(x)(-\sin(x))) \\
\frac{df}{dy}(x,y) & = \frac{1}{2}(y + \cos^2(x))^{-1/2}
\end{align*}
\]

We will find the linearization at \((0,0)\) since we can easily compute the following:

\[
\begin{align*}
\frac{df}{dx}(0,0) & = (0 + \cos(0))^{1/2} = 1 \\
\frac{df}{dy}(0,0) & = \frac{1}{2}(0 + \cos^2(0))^{-1/2} = 0 \\
\frac{df}{dt}(0,0) & = \frac{1}{2}(0 + \cos^2(0))^{-1/2} = \frac{1}{2}
\end{align*}
\]

Tangent Plane: 
\[
\begin{align*}
z - 0 & = 0(x-0) + \frac{1}{2}(y-0) \\
& = \frac{1}{2} y + 1
\end{align*}
\]

Thus, near \((0,0)\): 
\( f(x,y) \approx \frac{1}{2} y + 1, \), so 
\( f(0.2, 0.1) \approx \frac{1}{2}(0.1) + 1 = 1.05 \)
Directional Derivatives
Section 14.6

If $f$ is differentiable, then the **directional derivative** in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ is denoted $D_{\mathbf{u}}f(x, y)$ and is defined by:

$$D_{\mathbf{u}}f(x, y) = a f_x(x, y) + b f_y(x, y)$$

1. Find the directional derivatives of $f(x, y) = -x^2 + 2y^2$ at the point (1, 2):

   (a) in the direction of $\mathbf{v} = \langle 2, -4 \rangle$.

   Unit vector in the direction of $\mathbf{v}$:
   $$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 2, -4 \rangle}{\sqrt{2^2 + (-4)^2}} = \frac{\langle 2, -4 \rangle}{\sqrt{20}} = \frac{\langle 2, -4 \rangle}{2 \sqrt{5}}$$

   Directional Derivative:
   $$D_{\hat{\mathbf{v}}}f(1, 2) = \frac{2}{\sqrt{5}} f_x(1, 2) + \frac{-4}{\sqrt{5}} f_y(1, 2) = \frac{2}{\sqrt{5}} (-2) + \frac{-4}{\sqrt{5}} 8 = \frac{-18}{\sqrt{5}}$$

   (b) in the direction of $\theta = \frac{\pi}{4}$.

   Unit vector: $\hat{\mathbf{u}} = \frac{\langle 1, 1 \rangle}{\sqrt{2}} = \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

   Directional derivative:
   $$D_{\hat{\mathbf{u}}}f(1, 2) = \frac{\sqrt{2}}{2} f_x(1, 2) + \frac{\sqrt{2}}{2} f_y(1, 2) = \frac{\sqrt{2}}{2} (-2) + \frac{\sqrt{2}}{2} 8 = \frac{6}{\sqrt{2}}$$

2. Find the direction(s) in which the directional derivative of $f(x, y) = e^{-xy}$ at the point (0, 2) has value 1.

   Directional derivatives:
   $$f_x(x, y) = -y e^{-xy} \quad f_y(x, y) = -x e^{-xy}$$
   $$f_x(0, 2) = -2 e^0 = -2 \quad f_y(0, 2) = -0 e^0 = 0$$

   If $\mathbf{u} = \langle a, b \rangle$ is a unit vector, then
   $$D_{\mathbf{u}}f(0, 2) = a f_x(0, 2) + b f_y(0, 2) = a (-2) + b (0) = -2a$$

   Want: $-2a = 1$, so $a = \frac{-1}{2}$

   Unit vector: $a^2 + b^2 = 1$, so $\left(\frac{-1}{2}\right)^2 + b^2 = 1$, or $b = \frac{3}{4}$ and $b = \pm \frac{3}{4}$.
The gradient of $f(x, y)$ is a vector, denoted $\nabla f$, and defined by:

\[ \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \]

The maximum value of the directional derivative $D_uf$ is $|\nabla f|$, and it occurs when $u$ has the same direction as the gradient vector $\nabla f$.

3. Find the gradient of $f(x, y) = x^2 + x \sin(2y)$.

4. What is the direction of the maximum rate of change of $f(x, y) = x^2 + x \sin(2y)$ at the point $(2, 0)$? What is this rate of change?

5. Let $f(x, y)$ have continuous partial derivatives and consider the points $A(1, 2)$, $B(4, 2)$, $C(1, 5)$, and $D(5, 8)$. The directional derivative of $f$ at $A$ in the direction of vector $\overrightarrow{AB}$ is 3, and the directional derivative at $A$ in the direction of $\overrightarrow{AC}$ is 15. Find the directional derivative of $f$ at $A$ in the direction of vector $\overrightarrow{AD}$.

6. Explain why the directional derivative is really a dot product involving the gradient vector.