

# EXISTENCE AND UNIQUENESS

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

• Does a solution exist?

• Is it unique?

## THEOREM:

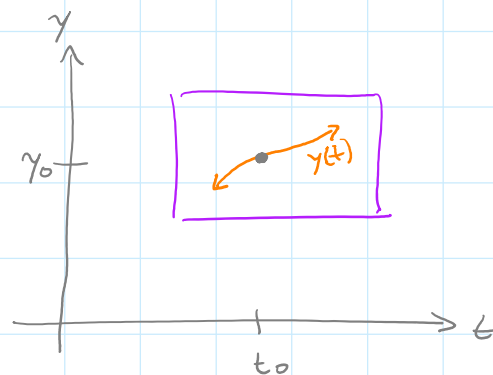
EXISTENCE

If  $f(t, y)$  is continuous in a rectangle containing  $(t_0, y_0)$ , then a solution exists through  $(t_0, y_0)$ .

UNIQUENESS

If  $f(t, y)$  and  $\frac{\partial f}{\partial y}$  are both continuous in a rectangle containing  $(t_0, y_0)$ , then the solution is unique near  $(t_0, y_0)$ .

differentiate  $f$  with respect to  $y$ , regard  $t$  as a constant



## WORKSHEET

Problems and solutions on following pages.

# Existence and Uniqueness

Math 230

1. Consider the differential equation  $\cos(t)y' - \sin(t)y = 3t \cos(t)$ .

(a) At what points  $(t, y)$  does a solution exist?

First, solve for  $y'$ :  $y' = \tan(t)y + 3t$ .

So  $f(t, y) = \tan(t)y + 3t$ , which is continuous whenever  $t \neq (n + \frac{1}{2})\pi$  for integer  $n$ . The existence theorem says that a solution exists at these points.

(b) At what points is the solution unique?

Since  $\frac{\partial f}{\partial y} = \tan(t)$ , which is continuous whenever  $t \neq (n + \frac{1}{2})\pi$  for integer  $n$ , the uniqueness theorem says that the solution at these points is unique.

(c) If a solution  $y(t)$  is such that  $y(0) = 0$ , what is the largest interval on which this solution is unique?

The solution is unique on the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

2. Consider the autonomous differential equation  $\frac{dy}{dt} = |y|$ .

(a) What are the equilibrium solutions?

The only equilibrium solution is  $y = 0$ .

(b) For what values of  $y$  does a solution exist?

Since  $f(t, y) = |y|$  is continuous everywhere, a solution exists at any point  $(t, y)$ .

(c) For what values of  $y$  is there a unique solution?

$$\frac{\partial f}{\partial y} = \begin{cases} -1 & \text{if } y < 0 \\ 1 & \text{if } y > 0 \end{cases}$$

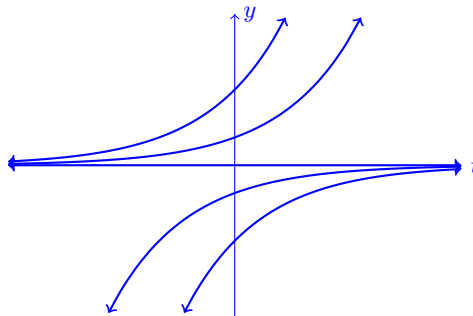
Since  $\frac{\partial f}{\partial y}$  is continuous when  $y \neq 0$ , the uniqueness theorem guarantees that the solution is unique wherever  $y \neq 0$ .

(d) Find all solutions, and sketch the family of solutions. *Hint:* Consider the cases  $y > 0$  and  $y < 0$  separately, and separate variables. Then consider the case  $y = 0$ .

If  $y > 0$ , then  $\frac{dy}{dt} = y$ , and we can separate variables to obtain  $y = Ke^t$  for some constant  $K > 0$ .

If  $y < 0$ , then  $|y| = -y$ , so  $\frac{dy}{dt} = -y$ . Separating variables gives  $\frac{dy}{y} = -dt$ , and we integrate to obtain  $\ln |y| = -t + C$ . Since  $y$  is negative,  $\ln |y| = \ln(-y)$ , and we obtain  $y = Ke^{-t}$  for some constant  $K < 0$ .

Together with the equilibrium solution  $y = 0$ , we the family of solutions looks like this:



3. Suppose  $f(t, y)$  satisfies the hypotheses of the existence and uniqueness theorem for all  $(t, y)$ . Also suppose that  $y_1(t) = 3$ ,  $y_2(t) = 6$ , and  $y_3(t) = t^2 + 8$  are solutions to  $\frac{dy}{dt} = f(t, y)$  for all  $t$ . What can you say about solutions satisfying the following initial conditions?

- (a)  $y(0) = 4$   
 $3 < y(t) < 6$  for all  $t$
- (b)  $y(0) = 7$   
 $6 < y(t) < t^2 + 8$  for all  $t$
- (c)  $y(0) = 9$   
 $t^2 + 8 < y(t)$  for all  $t$

4. Consider the autonomous differential equation  $\frac{dy}{dt} = \frac{1}{(1+y)^2}$ .

- (a) For what values of  $y$  is there a unique solution?

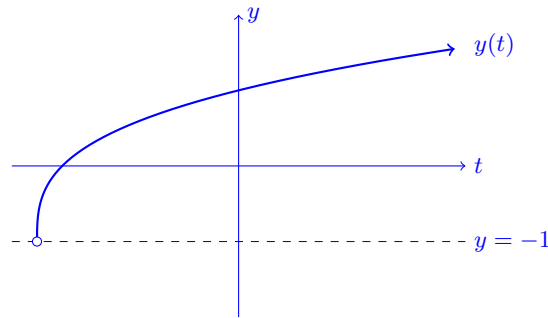
Since  $f(t, y) = (1 + y)^{-2}$  and  $\frac{\partial f}{\partial y} = -2(1 + y)^{-3}$  are continuous whenever  $y \neq -1$ , a unique solution exists at any point  $(t, y)$  with  $y \neq -1$ .

- (b) Find all solutions.

There are no equilibrium solutions. By separating variables, we find that the general solution is  $y(t) = \sqrt[3]{3t + C} - 1$  for some constant  $C$ .

- (c) If a solution  $y(t)$  is such that  $y(0) = 1$ , what is the largest interval on which this solution exists? Sketch the solution.

The particular solution is  $y(t) = \sqrt[3]{3t + 8} - 1$ , which exists on the interval  $(-\frac{8}{3}, \infty)$ . A sketch of this solution is:



5. Consider the autonomous differential equation  $\frac{dy}{dt} = 1 + y^2$ .

- (a) For what values of  $y$  is there a unique solution?

Since  $f(t, y) = 1 + y^2$  and  $\frac{\partial f}{\partial y} = 2y$  are continuous everywhere, a unique solution exists at any point  $(t, y)$ .

- (b) Find all solutions.

There are no equilibrium solutions. By separating variables, we find that the general solution is  $y = \tan(t + C)$  for some constant  $C$ .

- (c) If a solution  $y(t)$  is such that  $y(0) = 0$ , what is the largest interval on which this solution exists? *Hint:* What is the domain of the solution? Sketch the solution.

The particular solution is  $y(t) = \tan(t)$ , and the largest interval on which this solution is defined is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .