Families of Linear Systems Lab

Math 230

due Friday, November 9 at 4pm

In this lab you will investigate four different two-parameter families of linear systems of differential equations. In each case the parameters are two real numbers, a and b. Your goal is to provide a picture of the "parameter plane" (i.e., the ab-plane) for each family of linear systems. *Note*: We are not talking about the trace-determinant plane here; the horizontal axis should be the a-axis, and the vertical axis should be the b-axis.

For each family, first write the matrix representation of the system and find all eigenvalues (which, of course, depend on both a and b). Then determine the sets of points (a, b) for which there are repeated eigenvalues, zero eigenvalues, or purely imaginary eigenvalues, and draw (i.e. graph) these sets in the ab-plane.

Next, in each region between the curves that you graphed, determine the type of phase portrait that occurs (e.g., saddle, spiral sink, etc.). Also indicate what points (a, b) result in repeated zero eigenvalues.

You need not display the direction that solutions travel around the origin when the eigenvalues are complex, and you do not need to compute or display eigenvectors.

Finally, for each family, write a paragraph addressing the following: As the values of a and b change so that the point (a, b) moves from one region to another, the type of linear system changes—that is, a bifurcation occurs. At which bifurcations is there a change in the long-term behavior of solutions? How does the phase portrait change as the bifurcation occurs?

Here are the four families:

Family 1:

$$\frac{dx}{dt} = ay$$
$$\frac{dy}{dt} = bx$$

Family 2:

$$\frac{dx}{dt} = ax - by$$
$$\frac{dy}{dt} = \frac{b}{4}x$$

Family 3:

$$\frac{dx}{dt} = 2bx + (a+1)y$$
$$\frac{dy}{dt} = (a-1)x$$

Family 4:

$$\frac{dx}{dt} = 2bx - y$$
$$\frac{dy}{dt} = (\sin^2 a)x$$

Technology

To help understand the effect of the parameters a and b on the phase plane, consider using the `Manipulate` function in *Mathematica*. For example, try the following for Family 2:

```
Manipulate[ StreamPlot[ {a*x - b*y, b*x/4}, {x, -5, 5}, {y, -5, 5} ],
            {a, -20, 20}, {b, -20, 20} ]
```

Note that exploring the stream plot using `Manipulate` will not replace the algebraic analysis of the eigenvalues, but it can help you confirm that your analysis is correct.

Lab Report

Type your answers to the numbered items above in a document. You may use LaTeX or a word processor. You may draw diagrams by hand. *Do not* turn in a Mathematica file. If you submit your lab electronically, please submit a **PDF** file.

Grading

Your lab report will be graded out of 40 points total. Each family will be graded out of 9 points, as follows:

- Correct identification of the eigenvalues, in terms of a and b. (2 points)
- Correct identification of all bifurcation curves. (3 points)
- Accurate sketch of the bifurcation curves in the ab-plane, with a correct representative phase portrait in each region bounded by these curves. (2 points)
- Accurate description of each bifurcation. (2 points)

The remaining 4 points will be awarded for clarity and presentation.

- **Clarity**: Explanations are clear and concise. English sentences are used along with appropriate mathematical details to explain your reasoning. (2 points)
- **Presentation**: Work is presented in a typed document that is neat, organized, and easy to read. (2 points)

based on a lab by Robert L. Devaney: `http://math.bu.edu/people/bob/MA226/lin-sys-family-lab.html`