TRACE-DETERMINANT PLANE

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$T = tr(\mathbf{A})$

$D = det(\mathbf{A})$

- Characteristic Polynomial

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - \underbrace{(a+d)}_{\text{trace } T}\lambda + \underbrace{(ad-bc)}_{\text{determinant } D} = \lambda^2 - T\lambda + D$$

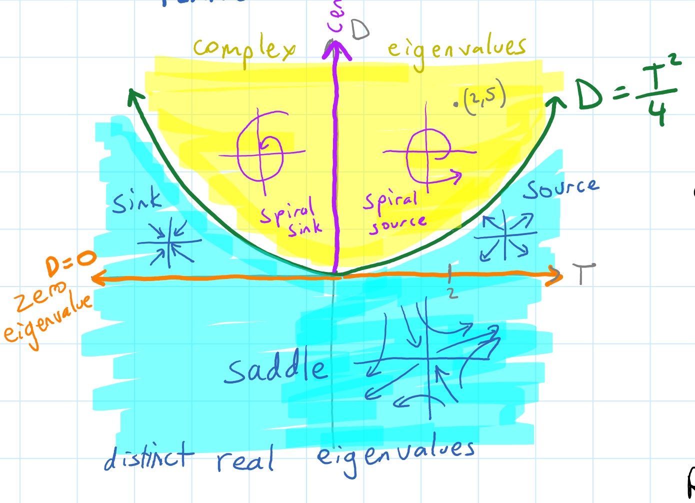
- Eigenvalues:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Observe:

1. If $T^2 - 4D > 0$, then two real eigenvalues.
2. If $T^2 - 4D = 0$, then repeated real eigenvalue. ⇦ bifurcation curve $T^2 = 4D$, $D = \frac{T^2}{4}$
3. If $\underbrace{T^2 - 4D < 0}_{\to D > \frac{T^2}{4}}$, then complex eigenvalues.

TRACE-DETERMINANT PLANE



REPEATED-ROOT PARABOLA

Complex eigenvalues: $\lambda = \underbrace{\frac{T}{2}}_{\text{real}} \pm i \underbrace{\frac{\sqrt{4D - T^2}}{2}}_{\text{imaginary}}$

- If $\frac{T}{2} > 0$, then spiral source.
- If $\frac{T}{2} = 0$, then center.
- If $\frac{T}{2} < 0$, then spiral sink.

Real eigenvalues: $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

- If $\sqrt{T^2 - 4D} > T$, then eigenvalues have different signs.

  $T^2 - 4D > T^2$

  $-4D > 0$

  $D < 0$

  If $D < 0$, then eigenvalues have opposite signs. } Saddle

- If $D = 0$, then $\lambda = \frac{T \pm \sqrt{T^2}}{2} = \frac{T \pm T}{2}$, $\lambda = 0$, $\lambda = T$

  zero eigenvalue

- If $D > 0$, then eigenvalues have same sign, the same sign as T:

  $T > 0$: source

  $T < 0$: sink

EXAMPLE:

Matrix $\mathbf{A}$ has trace 2, and determinant 5.

Then phase portrait of $\frac{dY}{dt} = \mathbf{A}Y$ is a spiral source.

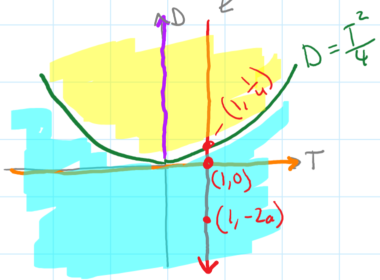
WORKSHEET

1. $A = \begin{bmatrix} 0 & a \\ 2 & 1 \end{bmatrix}$

Trace: $T = 1$

Det: $D = -2a$

(a)



(b) $D = \frac{T^2}{4} \Rightarrow -2a = \frac{1^2}{4}$

$-8a = 1$

$a = -\frac{1}{8}$

If $a = -\frac{1}{8}$, then $D = \frac{T^2}{4}$, so we have a repeated root

If $a = 0$, then zero eigenvalue.

$a < -\frac{1}{8}$ | $a = -\frac{1}{8}$ | $-\frac{1}{8} < a < 0$ | $a = 0$ | $a > 0$