## Math 234

Sequences Day 9

Discuss the following problems with the people at your table.

1. A sequence is defined by $a_k = \frac{k}{2k+6}$. Write the terms $a_1, a_2, a_3$, and $a_4$.

$a_1 = \frac{1}{8}$, $a_2 = \frac{2}{10} = \frac{1}{5}$, $a_3 = \frac{3}{12} = \frac{1}{4}$, $a_4 = \frac{4}{14} = \frac{2}{7}$

2. Find an explicit formula for each sequence $a_1, a_2, a_3, \ldots$ below.

(a) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

$$a_k = \frac{1}{k^2}$$

(b) $\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \frac{10}{6}, \ldots$

$$a_k = \frac{2k}{k+1}$$

same!

(c) $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \ldots$

$$a_k = \frac{2k}{k+1} = \frac{2(k+1) - 2}{k+1}$$

(d) $\frac{-1}{3}, \frac{1}{6}, \frac{-1}{11}, \frac{1}{18}, \frac{-1}{27}, \ldots$

$$a_k = (-1)^k \frac{1}{k^2+2}$$

parentheses are important!

$$\frac{(-1)^k}{k^2+2}$$

(e) $\frac{1}{3}, \frac{-2}{7}, \frac{3}{13}, \frac{-4}{21}, \frac{5}{31}, \ldots$

$$a_k = \frac{k(-1)^{k+1}}{k^2+k+1} = \frac{k(-1)^{k-1}}{k(k+1)+1}$$

3. Compute the sum $\sum_{m=0}^{3} \frac{1}{2^m}$.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$$

4. Compute the product $\prod_{k=1}^{3} \left(1 + \frac{1}{k}\right)$

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) = 4$$

5. Write the following in summation notation:

$$(1^5 - 1) + (2^5 - 1) + (3^5 - 1) + (4^5 - 1) + (5^5 - 1)$$

$$\sum_{m=1}^{5} (m^5 - 1)$$

6. Write the following in product notation:

$$\left(\frac{2}{4}\right)\left(\frac{3}{5}\right)^2\left(\frac{4}{6}\right)^3\left(\frac{5}{7}\right)^4$$

$$\prod_{k=2}^{5} \left(\frac{k}{k+2}\right)^{k-1} = \prod_{k=1}^{4} \left(\frac{k+1}{k+3}\right)^{k}$$

7. Transform the following by making the change of variables $\underline{j = i - 1}$

If $i = 1$, then $j = 0$.

If $i = n-1$, then $j = n-2$

$$\sum_{i=1}^{n-1} \frac{1}{(n-i)^2}$$

$j + 1 = i$

$$\sum_{j=0}^{n-2} \frac{1}{(n-(j+1))^2}$$

8. Transform the following by making the change of variables $k = i + 1$

If $i=0$, then $k=1$.

If $i=n$, then $k=n+1$.

$$\sum_{i=0}^{n} \frac{i}{i^2+1} = 0 + \frac{1}{2} + \dots + \frac{n}{n^2+1}$$

$$\sum_{k=1}^{n+1} \frac{k-1}{(k-1)^2+1} = \frac{0}{1} + \frac{1}{2} + \dots + \frac{n}{n^2+1}$$

9. Simplify the expressions:

(a) $\frac{100!}{98!} = \frac{(100)(99)\cancel{(98)}\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}}{\cancel{(98)}\cancel{(97)}\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}} = 100(99)$

$n! = (n)(n-1)\cdots(3)(2)(1)$

(b) $\frac{n!}{(n-3)!} = n(n-1)(n-2)$

(c) $\frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-(k-1))$

10. Compute the value of the combinations:

$$\boxed{\text{"n choose k"} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$$

(a) $\binom{10}{8} = \frac{10!}{8!\,2!} = \frac{10(9)}{2} = 45$

(b) $\binom{n}{n-2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$

assume $n \geq 2$

11. **Bonus:**

problem 66 in section 5.1 of the textbook

(a) Prove that $n! + 2$ is even for all integers $n \geq 2$.

(b) Prove that $n! + k$ is divisible by $k$ for all integers $n \geq 2$ and $k \in \{2, 3, \dots, n\}$.

(c) Given any integer $m \geq 2$, does there exist a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain.