# Math 234

Permutations and Binomial Coefficients — Day 17

1. How many different ways can the letters in the following words be arranged?

   (a) BOOKKEEPER

   ☞ triple-double letters!

   (b) UNSUCCESSFULLY

   (c) POSSESSIVENESS

   ☞ Can you think of a word with more s's?

2. Compute $\binom{4}{k}$ for each $k \in \{0, 1, 2, 3, 4\}$.

3. Draw a copy of Pascal's triangle, at least through the row corresponding to $n = 4$. How does this relate to your answers to #2?

4. Use Pascal's formula (several times) to derive a formula for $\binom{n+3}{r}$ in terms of values of $\binom{n}{k}$ with $k \leq r$.

☞ You may assume $n$ and $r$ are integers with $n \geq r \geq 3$.

5. Expand $(x+y)^5$. How are combinations involved here?

6. (a) Verify each of the following identities:

$$3^2 = 2^2 + 2 \cdot 2 + 1$$

$$3^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$$

$$3^4 = 2^4 + 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$

☞ Do you see the binomial coefficients?

$$3^5 = 2^5 + 5 \cdot 2^4 + 10 \cdot 2^3 + 10 \cdot 2^2 + 5 \cdot 2 + 1$$

(b) The verifications in part (a) seem to suggest the following identity:

$$3^n = 2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} \cdots + \binom{n}{n-1}2 + 1$$

Express this identity using summation notation, and show how it follows from the Binomial Theorem.

7. Prove that $\sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i} = 2^n$ for all integers $n \geq 0$.

8. Use the binomial theorem to evaluate the sum: spicy!

$$\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{2^2}\binom{n}{2} - \frac{1}{2^3}\binom{n}{3} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}$$

9. **BONUS:**

(a) Choose several rows of Pascal's triangle. Add up the numbers in each row. What do you notice?

(b) Based on your observations in part (a), what do you think $\sum_{k=0}^{n} \binom{n}{k}$ equals?

(c) How does your answer to part (c) relate to the power set of a set of $n$ elements?