MATH 242 — 22 Sept. 2025

Fibonacci Sequence $F_0 = 0,\ F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ for $n > 1$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Proof that $\displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$, assuming that the limit exists.

Suppose that $\displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists. Say $\displaystyle\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = x$.

Recall the Fibonacci recurrence: $F_n = F_{n-1} + F_{n-2}$.

Divide by F_{n-1}:

$$\frac{F_n}{F_{n-1}} = \frac{F_{n-1}}{F_{n-1}} + \frac{F_{n-2}}{F_{n-1}}$$

Take limit as $n \to \infty$.

$$\lim_{n\to\infty}\left(\frac{F_n}{F_{n-1}}\right) = \lim_{n\to\infty} 1 + \frac{1}{\lim_{n\to\infty}\left(\frac{F_{n-1}}{F_{n-2}}\right)}$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}} = x$$

$$x = 1 + \frac{1}{x}$$

$$\boxed{x^2 = x + 1} \leftarrow \phi^2 = \phi + 1$$

$$\underbrace{x^2 - 1x - 1 = 0}$$

Apply the quadratic formula: $\displaystyle x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Since $F_n > 0$ for $n > 1$, we know $x > 0$.

So, take the + case: $\displaystyle x = \frac{1+\sqrt{5}}{2}$.

$$\lim_{n\to\infty} \frac{F_n}{F_{n-k}} + \lim_{n\to\infty} \frac{F_n}{F_{n-(k-1)}} = \lim_{n\to\infty} \frac{F}{F_{n-(k+1)}}$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-2}} = 2.61803\ldots = \phi + 1 = \phi^2$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-3}} = \phi^3$$

$$\vdots$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-k}} = \phi^k$$

$$F_{n-1} F_{n+1} + (-1)^n = F_n^2 \quad ?$$

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n \quad ?$$