MATH 242 — 26 September 2025

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2\ 3, 5, 8, 13, 21, \dots$

Question: Could there exist a Fibonacci identity of the form

$$F_{3n} = \underline{a F_n^3 + b F_n^2 + c F_n} \ ?$$

Here, $\underline{a, b, c}$ are some numbers to be determined.

3 unknown coeffients — set up and solve 3 equations

Example equation: $n = 1$:

$$F_3 = a F_1^3 + b F_1^2 + c F_1$$
$$2 = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1$$
$$\boxed{2 = a + b + c}$$

CONJECTURE: The Fibonacci numbers satisfy

$$F_{3n} = 5 F_n^3 + 3(-1)^n F_n$$

for positive indexes n.

Fibonacci $3n$ Identity

$$\begin{bmatrix} 2 \\ 34 \\ 610 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 125 & 25 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{v} = M \vec{x}$$

$$M^{-1} \cdot \vec{v} = \vec{x}$$

Even n:

$$F_{3n} = 5 F_n^3 + 3 F_n$$

Fibonacci $5n$ identity:

$$F_{5n} = 25 F_n^5 + (-1)^n 25 F_n^3 + 5 F_n$$