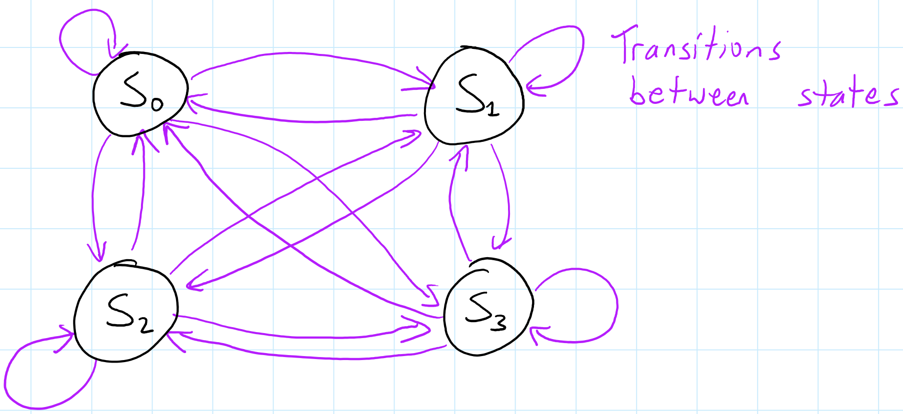
MARKOV CHAINS

example: 4 states



STEADY-STATE DISTRIBUTION: long-term distribution of how frequently each state occurs

3 PERSPECTIVES:

① SIMULATION:

② MATRIX POWERS Transition matrix P

Take a high power: P^n then the columns of P^n give the steady-state distribution

③ EIGENVECTOR: If $\vec{v}$ is the steady-state dist., then $P\vec{v} = \vec{v}$, which means that $\vec{v}$ is an eigenvector of P corresponding to eigenvalue 1.

INVERSE PROBLEM:

Given a steady-state vector, find a Markov chain.

Example: Let $\vec{p} = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$. Find a Markov Chain whose steady-state distribution is $\vec{p}$.

$$P_{0,1} = \frac{p_0}{p_1} Q_{1,0} = \frac{\frac{1}{6}}{\frac{1}{3}} \cdot \frac{1}{3} = \frac{1}{6}$$

Why does the procedure work? ($n=3$ case)

Let $\vec{p} = (p_0, p_1, p_2)$ be the steady-state vector.

WLOG, assume $p_0 \le p_1 \le p_2$ (renumber states if necessary).

Let Q be a matrix with each column sum equal to 1.

Then: Since p_0 is smallest: $P_{1,0} = Q_{1,0}$, $P_{2,0} = Q_{2,0}$, $P_{0,0} = 1 - Q_{1,0} - Q_{2,0}$

p_1 is next smallest, so: $P_{2,1} = Q_{2,1}$, $P_{0,1} = \frac{p_0}{p_1} Q_{1,0}$, $P_{1,1} = 1 - Q_{2,1} - \frac{p_0}{p_1} Q_{1,0}$

p_2 is largest: $P_{0,2} = \frac{p_0}{p_2} Q_{2,0}$, $P_{1,2} = \frac{p_1}{p_2} Q_{2,1}$, $P_{2,2} = 1 - \frac{p_0}{p_2} Q_{2,0} - \frac{p_1}{p_2} Q_{2,1}$

Thus:

$$P\vec{p} = \begin{bmatrix} 1 - Q_{1,0} - Q_{2,0} & \frac{p_0}{p_1} Q_{1,0} & \frac{p_0}{p_2} Q_{2,0} \\ Q_{1,0} & 1 - Q_{2,1} - \frac{p_0}{p_1} Q_{1,0} & \frac{p_1}{p_2} Q_{2,1} \\ Q_{2,0} & Q_{2,1} & 1 - \frac{p_0}{p_2} Q_{2,0} - \frac{p_1}{p_2} Q_{2,1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

$$= \begin{bmatrix} p_0 - \cancel{p_0 Q_{1,0}} - \cancel{p_0 Q_{2,0}} + \cancel{p_0 Q_{1,0}} + \cancel{p_0 Q_{2,0}} \\ \cancel{p_0 Q_{1,0}} + p_1 - \cancel{p_1 Q_{2,1}} - \cancel{p_0 Q_{1,0}} + \cancel{p_1 Q_{2,1}} \\ p_0 Q_{2,0} + p_1 Q_{2,1} + p_2 - p_0 Q_{2,0} - p_1 Q_{2,1} \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \vec{p}$$