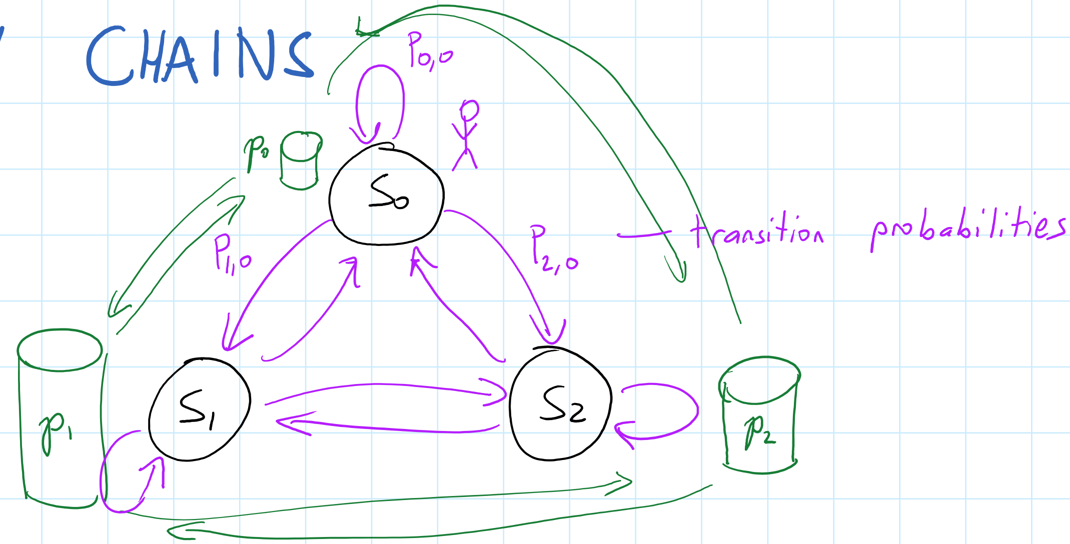
MARKOV CHAINS

States
S_0, S_1, S_2



STEADY-STATE DISTRIBUTION

- Long-term distribution of states visited
- Distribution of stuff that remains the same from one time step to the next.
- Power of the transition matrix: P^n

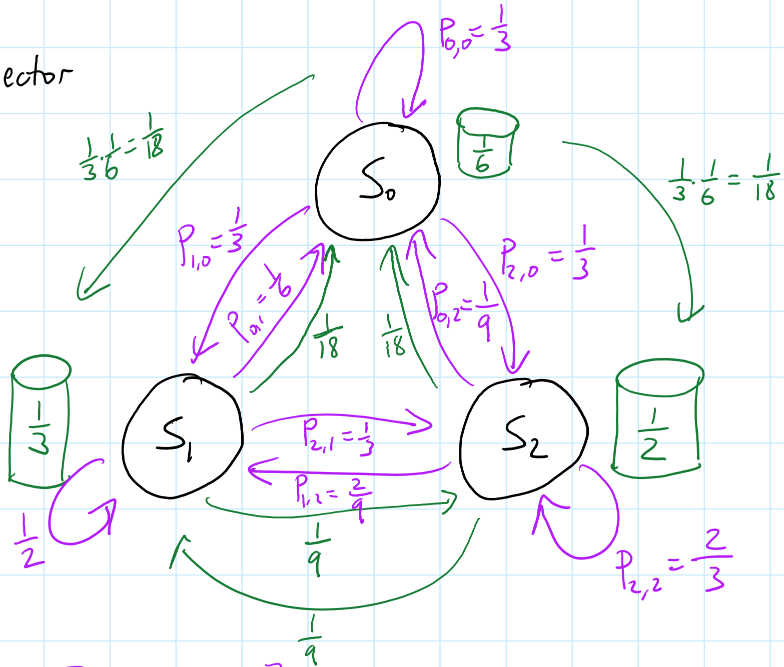
  If n is big, then each column of P^n gives the steady-state distribution.

- Transition matrix P has an eigenvector v corresponding to eigenvalue 1: $P\vec{v} = \vec{v}$ $\quad$ $\vec{v}$ is the steady-state distribution

INVERSE PROBLEM: Given a steady-state vector, find a Markov chain with that steady-state vector.

EXAMPLE: Steady-state vector

$\vec{p} = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$



$\frac{1}{3} P_{0,1} = \frac{1}{3}\cdot\frac{1}{6}$

$P_{0,1} = \frac{1}{6}$

Transition Matrix: $P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ ← STOCHASTIC MATRIX

Why does the method work?

$n = 3$

Let $\vec{p} = (p_0, p_1, p_2)$. WLOG, $p_0 \le p_1 \le p_2$. (Re-index states if necessary).

Choose matrix Q of non-negative entries, column sums 1.

Then:

$$P = \begin{bmatrix} 1 - Q_{1,0} - Q_{2,0} & \frac{p_0}{p_1} Q_{1,0} & \frac{p_0}{p_2} Q_{2,0} \\ Q_{1,0} & 1 - \frac{p_0}{p_1} Q_{1,0} - Q_{2,1} & \frac{p_1}{p_2} Q_{2,1} \\ Q_{2,0} & Q_{2,1} & 1 - \frac{p_0}{p_2} Q_{2,0} - \frac{p_1}{p_2} Q_{2,1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

$$P\vec{p} = \begin{bmatrix} p_0 \cancel{- p_0 Q_{1,0}} \cancel{- p_0 Q_{2,0}} \cancel{+ p_1 \frac{p_0}{p_1} Q_{1,0}} \cancel{+ p_2 \frac{p_0}{p_2} Q_{2,0}} \\ \text{etc.} \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \vec{p}$$