Wednesday: Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

investigated $\frac{F_n}{F_{n-1}}$, we saw that $\frac{F_n}{F_{n-1}} \to 1.618$ as $n \to \infty$

Why? Assume that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists.

Consider the recursive definition $F_n = F_{n-1} + F_{n-2}$.

Divide by F_{n-1}: $$\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

Take the limit: $$\boxed{\lim_{n\to\infty} \frac{F_n}{F_{n-1}}} = 1 + \boxed{\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}}$$

$$\lim_{n\to\infty}\left(\frac{1}{\frac{F_{n-1}}{F_{n-2}}}\right) = \frac{1}{\lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}}$$

Let $x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$: $$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$ ← quadratic equation

$$x^2 - x - 1 = 0$$

Quadratic Formula: $$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since we know F_n is positive, $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ must be the positive root $\frac{1+\sqrt{5}}{2} = \varphi \approx 1.618\ldots$ "golden ratio"

Investigate: $$\underbrace{F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}}_{\text{Cassini's identity}}$$ for various n

Verification using lists:

$$F_1^2 - F_2 F_0 = 1$$
$$F_2^2 - F_3 F_1 = -1$$
$$F_3^2 - F_4 F_2 = 1$$
$$F_4^2 - F_5 F_3 = -1$$
$$\vdots$$
$$F_{19}^2 - F_{20} F_{18} = 1$$

fibSquared fibA fibB fibSlice 2

fibSquared
list

fibA
fibSlice 1

fibB
fibSlice 2

Mathematica: fibSquared − fibA * fibB

Verification using a module:

Write a module that checks $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$ for a particular n. Then call the module many times.

Complete the identity: $F_n^2 - F_{n+r}F_{n-r} =$ ___?___