Wednesday: Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

Investigate $\frac{F_n}{F_{n-1}}$ as n gets big. $\frac{F_n}{F_{n-1}} \approx 1.618$

Why? Assume the limit exists: $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists.

Start with recursive def: $F_n = F_{n-1} + F_{n-2}$

Divide by F_{n-1}: $\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$

Take the limit: $\boxed{\lim_{n\to\infty} \frac{F_n}{F_{n-1}}} = 1 + \boxed{\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}}$

$\lim_{n\to\infty} \left(\frac{1}{\frac{F_{n-1}}{F_{n-2}}} \right) = \frac{1}{\lim \left(\frac{F_{n-1}}{F_{n-2}}\right)} = \frac{1}{x}$

Let $x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$: $\quad x = 1 + \frac{1}{x}$

$x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}$

$x^2 = x + 1$ ← quadratic equation

$x^2 - x - 1 = 0$

Use quadratic formula: $x = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

We know $\frac{F_n}{F_{n-1}}$ is positive, so take the positive root:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} = \varphi \approx 1.618$$

↑ Golden Ratio

Investigate: $\underbrace{F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}}_{\text{Cassini's Identity}}$

Verification #1: using lists

$n=2$: $F_2^2 - F_3 F_1 = -1$

$n=3$: $F_3^2 - F_4 F_2 = 1$

$n=4$: $F_4^2 - F_5 F_3 = -1$

$n=5$: $F_5^2 - F_6 F_4 = 1$

$\vdots$

$n=19$: $F_{19}^2 - F_{20} F_{18} = 1$ $\quad m=20$

n=19 $\underbrace{F_{19}^2}_{\text{fibSquared}} - \underbrace{\underbrace{F_{20}}_{\text{fibSlice1}}\,\underbrace{F_{18}}_{\text{fibSlice2}}}_{\substack{\text{multiply} \\ \text{fibProduct}}} = 1$ m=20

Verification #2: write a module that checks $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ for a specific n, then call it lots of times for different n

Investigate: $F_n^2 - F_{n+r}F_{n-r} =$ ___?___