MATH 242: Wednesday, March 18, 2020

PRIME CONJECTURES

Goldbach's conjecture: probably true

- **Conjecture A.** Every even integer greater than 2 is the sum of two primes.
- **Conjecture B.** For any N, the number of nonnegative integers less than N with an *even* number of prime factors is less than the number of nonnegative integers less than N with an *odd* number of prime factors. For this, prime factors are counted *with multiplicity*; e.g., $24 = 2^3 \cdot 3$ has 4 prime factors, while $588 = 2^2 \cdot 3 \cdot 7^2$ has 5 prime factors. FALSE: counterexample 906,150,257

Polya conjecture

- **Conjecture C.** For any positive integer n, there exists at least one prime between n^2 and $(n+1)^2$. – no counterexample known
- **Conjecture D.** All odd numbers greater than 1 are either prime, or can be expressed as the sum of a prime and twice a square. FALSE: counterexamples 5777 and 5993

PRIME COUNTING FUNCTION

$\pi(x)$ = number of primes less than or equal to x

EXAMPLE: $\pi(8) = 4$, since $2, 3, 5, 7 \le 8$

IMPLEMENTATION: countPrimes[n] returns $\{\pi(1), \pi(2), \pi(3), \ldots, \pi(n)\}$

$$\text{primeList: } \{\underset{1}{2}, \underset{2}{3}, \underset{3}{5}, \underset{4}{7}, \underset{5}{11}, \underset{6}{13}, \underset{7}{17}, \underset{8}{19}, \underset{9}{23}, \underset{\ldots}{\ldots}, P\}$$

$c=4$

values of $\pi(1), \ldots, \pi(n)$ →

$$\text{counts: } \{\underset{1}{\overset{\pi(1)}{0}}, \underset{2}{\overset{\pi(2)}{1}}, \underset{3}{\overset{\pi(3)}{2}}, \underset{4}{\overset{\pi(4)}{2}}, \underset{5}{\overset{\pi(5)}{3}}, \underset{6}{\overset{\pi(6)}{3}}, \underset{7}{\overset{\pi(7)}{4}}, \underset{8}{4}, \underset{9}{4}, \underset{10}{4}, \underset{11}{5}, \ldots \underset{n}{0}\}$$

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Exercise 1: Plot $\pi(x)$. Use ListPlot.

What is the shape of the graph?

DENSITY OF PRIMES

number of primes

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number of primes $x < p \le x+d$

Choose an interval length d, and a number x.

Density of primes near x is approximately $\dfrac{\boxed{\pi(x+d) - \pi(x)}}{d}$

EXAMPLE: $d = 10$, $x = 100$: $\dfrac{\pi(100+10) - \pi(100)}{10} = \dfrac{29 - 25}{10} = \dfrac{4}{10} = 0.4$

Gauss: density of primes is approximately $\dfrac{1}{\ln(x)}$

e.g. $\dfrac{1}{\ln(100)} = 0.217$

Exercise 2: Plot the prime density function $\dfrac{\pi(x+d) - \pi(x)}{d}$.

Do you think that Gauss' estimate $\boxed{\dfrac{1}{\ln(x)}}$ is good?

CONTINUOUS APPROX FOR PRIME COUNTING FUNCTION

simple approx: $\dfrac{x}{\ln(x)} \approx \pi(x)$

logarithmic integral: $\mathrm{li}(x) = \displaystyle\int_0^x \frac{dt}{\ln(t)}$

integrate

Exercise 3: Plot $\pi(x)$, $\dfrac{x}{\ln(x)}$, $\mathrm{li}(x)$ on the same plot.

Are these approximations good? What is the percent error?

TWIN PRIMES:

pair of primes $(p, p+2)$

EXAMPLES: $(5,7)$, $(11,13)$

Twin prime counting function: $\pi_2(x)$ = number of primes $p \le x$ such that $p+2$ is also prime

Exercise 4: Plot $\pi_2(x)$. What do you notice?

How does the graph $\pi_2(x)$ compare with $\pi(x)$.

OTHER PAIRS OF PRIMES:

Cousin primes: $(p, p+4)$ Sexy primes: $(p, p+6)$

counting function: $\pi_4(x)$ counted: $\pi_6(x)$

Exercise 5: Plot $\pi_2(x)$, $\pi_4(x)$, $\pi_6(x)$. What do you notice?