Math 242: Friday, March 20, 2020

Zeta Function: $\zeta(s) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^s}}$ if $s > 1$

p-series from calculus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

Exercise 1:

Compute $\zeta(s)$ for some s.

For example: $\zeta(2)$, $\zeta(3)$, $\zeta(4)$ ...

ANALYTIC CONTINUATION:

Technique to extend the domain of a function.

There exists a complex-valued function $\zeta(s)$ for any $s \in \mathbb{C}$, that agrees with $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for real $s > 1$.

Riemann Zeta Function: **Zeta[s]**

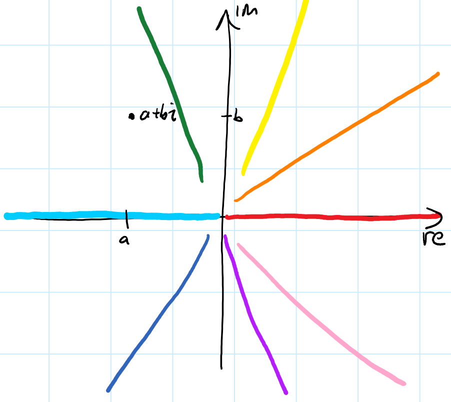
Exercise 2: $\zeta(0)$, $\zeta(-1)$, $\zeta(-2)$, $\zeta(-3)$, ...

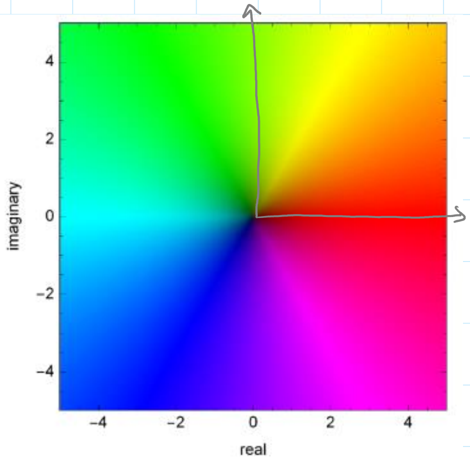
DOMAIN COLORING

complex numbers

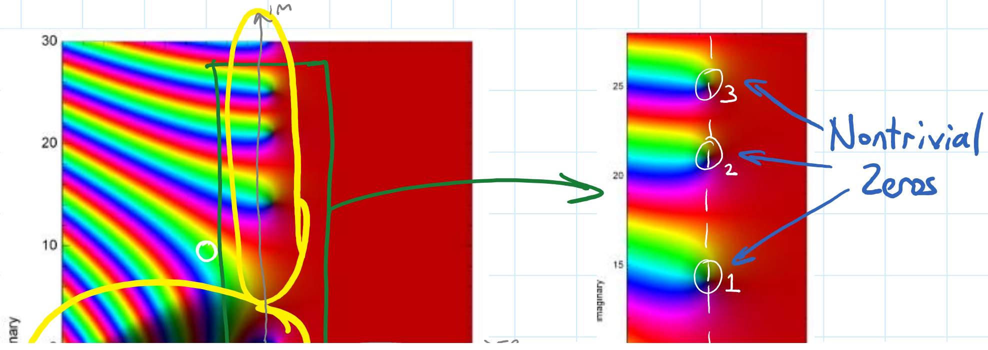
$s = a + bi$

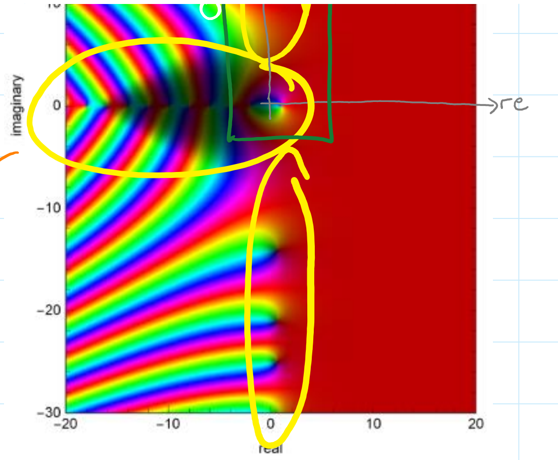
real part, imaginary part, $i^2 = -1$

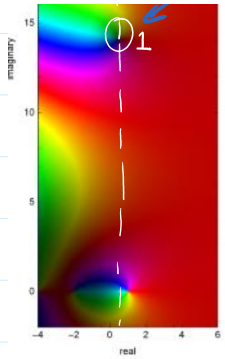




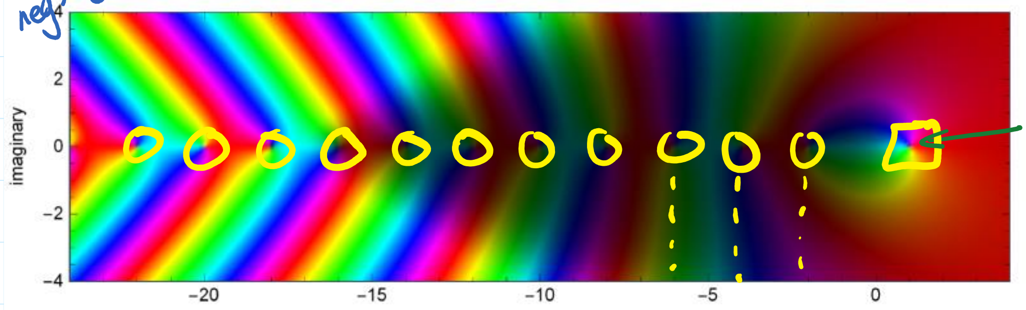
Riemann Zeta Function:







Trivial Zeros at neg. even int's.



$s = 1$ is the pole of $\zeta(s)$

RIEMANN HYPOTHESIS: All nontrivial zeros of the Riemann Zeta function lie on the line $\text{Re}(s) = \frac{1}{2}$.

Exercise 3: Make a domain coloring for $f(z) = \sin(z)$.

Mathematica: **Zeta Zero[k]** returns the k^{th} nontrivial zero of $\zeta(s)$

CONNECTIONS TO PRIMES

Euler product formula: $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \underbrace{\prod_{p \text{ prime}} \frac{1}{1-p^{-s}}}_{} \quad \text{for } s > 1$$

$$= \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\cdots$$

Exercise 4: Compute $\prod_{\substack{\text{primes} \\ p \in \{2,3,\dots,\_\}}} \left(\frac{1}{1-p^{-s}}\right)$ for various $s > 1$, verify that you get approximately $\zeta(s)$ in each case.

Prime counting: $\pi(x)$ = number of primes $\leq x$

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k! \; \zeta(k+1)}$$

← closely approximates $\pi(x)$

Second Chebyshev Function: $\psi(x) = \sum_{p^k \leq x} \ln(p)$ ← converges to

↑ all prime powers $\leq x$

Example: $\psi(5) = \ln(2) + \ln(2) + \ln(3) + \ln(5) \approx 4.09$

$2^1 \leq 5, \quad 2^2 \leq 5, \quad 3^1 \leq 5, \quad 5^1 \leq 5$

Let

$$\boxed{\psi_0(x) = x - \ln(2\pi) - \left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) - \frac{1}{2}\ln\left(1 - x^{-2}\right)}$$

↑ sum is over all nontrivial zeros of $\zeta(s)$

CONSEQUENCES OF THE RIEMANN HYPOTHESIS

EXERCISE 5: What have you learned about prime counting?
Interesting? Question?