23 Feb. 2024

Suppose that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists.

Start with the Fibonacci recurrence:

$$F_n = F_{n-1} + F_{n-2}$$

Divide by F_{n-1}:

$$\frac{F_n}{F_{n-1}} = \cancelto{1}{\frac{F_{n-1}}{F_{n-1}}} + \frac{F_{n-2}}{F_{n-1}}$$

Let $x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}$. Then:

$$\underbrace{\lim_{n\to\infty} \frac{F_n}{F_{n-1}}}_{x} = 1 + \frac{1}{\underbrace{\lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}}_{x}}$$

Thus:

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

By the quadratic formula: $x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Since $\frac{F_n}{F_{n-1}} > 0$, we choose the positive solution,

and thus $\boxed{x = \frac{1+\sqrt{5}}{2}.}$

GOLDEN RATIO

$\cancel{\frac{1-\sqrt{5}}{2} < 0}$

PROPERTY: $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$

Generalization: What about $\lim_{n\to\infty} \frac{F_n}{F_{n-2}}$

or $\frac{F_n}{F_{n-3}}$ or $\frac{F_n}{F_{n-k}}$?

$$\lim_{n\to\infty} \frac{F_n}{F_{n-k}} = \left(\frac{1+\sqrt{5}}{2}\right)^k$$

why? $\lim_{n\to\infty} \frac{F_n}{F_{n-2}} = \lim_{n\to\infty} \left(\frac{F_n}{\cancel{F_{n-1}}} \cdot \frac{\cancel{F_{n-1}}}{F_{n-2}}\right) = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \cdot \lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}$

$= \varphi \cdot \varphi$

Observe: $\varphi = \frac{1+\sqrt{5}}{2}$ satisfies

$$x^2 = x + 1$$

$$\varphi^2 = \varphi + 1$$

Thus:

$$\varphi^3 = \varphi^2 + \varphi$$

$$\varphi^4 = \varphi^3 + \varphi^2$$

$$\vdots$$

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observation

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n$$