26 Feb. 2024

0, 1, 1, 2, 3, 5, 8, ...

Last time, we observed $\underbrace{F_{n-1}F_{n+1} - F_n^2 = (-1)^n}_{\text{CASSINI'S IDENTITY}}$.

PROOF: Observe:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

proof by induction

For integers $n \geq 1$: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

inductive step

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix} = \begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}$$

Take determinants:

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \det \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

$$\left(\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^n = F_{n+1}F_{n-1} - F_n^2$$

Thus: $$\boxed{(-1)^n = F_{n+1}F_{n-1} - F_n^2}$$

for integers $n \geq 1$

INVESTIGATE:

$$F_{n-2}F_{n+2} - F_n^2 = ?$$

$$F_{n-3}F_{n+3} - F_n^2 = ?$$

$$F_{n-k}F_{n+k} - F_n^2 = ?$$

example:
try $k=5$: $F_{n-5}F_{n+5} - F_n^2 = ?$

$n=1$

F_{1-5}

F_{-4}

$n \geq k$