3 January 2025

Polygons and Triangulations

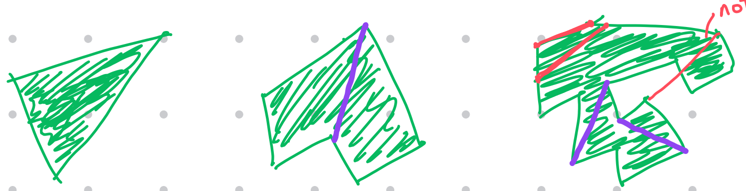
contains its boundary

POLYGON: A polygon is a closed region of the plane bounded by a finite collection of line segments forming a closed curve that does not intersect itself.

wraps around back to itself

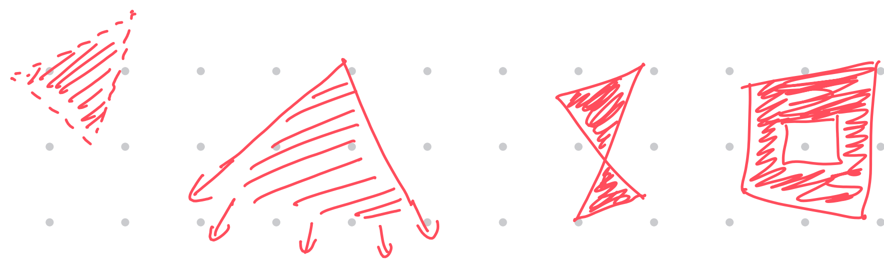
examples:

not a diagonal



examples of diagonals

not polygons:



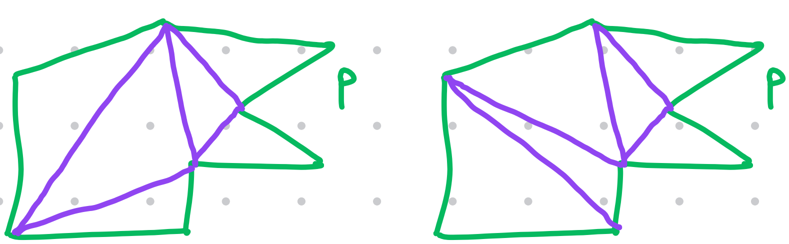
Let P denote a polygon.

DIAGONAL: A diagonal is a line segment connecting two vertices of P, lying in the interior of P, and only touching the boundary of P at its endpoints.

TRIANGULATION: A decomposition of P into triangles by a maximal set of non-crossing diagonals.

you can't add any more.

example:



Polygon Triangulation

MATH 261 Computational Geometry

Explore the following questions:

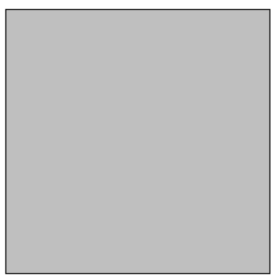
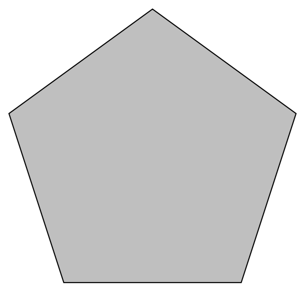
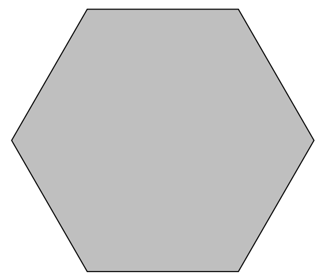
1. Does every polygon have a triangulation?

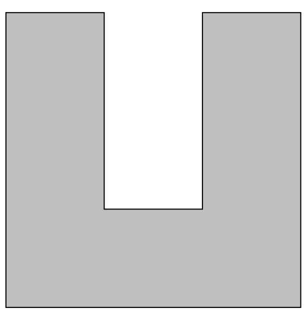
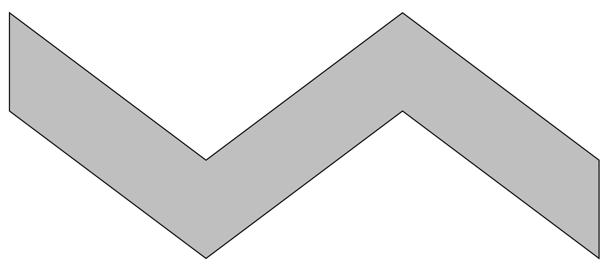
2. If P is a polygon with n vertices, how many triangles are required to triangulate P?

3. How many distinct triangulations are possible for a polygon of n vertices?

State your answers as conjectures, supported by your observations. If you can prove your conjecture, then you have a theorem.

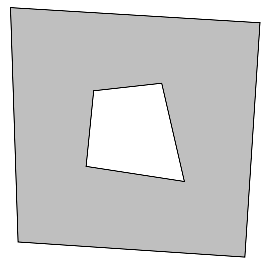
Here are some sample polygons. You should also consider other polygons of your choice.

Extension: Now consider polygons that are allowed to have holes. How do your answers to the previous questions change?

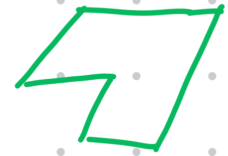
For example, the following polygon-with-hole has 8 vertices and 1 hole.



Does every polygon have a diagonal?

triangles — no

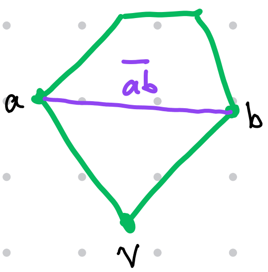
n-gons with $n > 3$ — yes!



LEMMA 1.3: Every polygon P with more than 3 vertices has a diagonal.

little theorem, used for proving something else

PROOF: We will use a line-sweep argument.



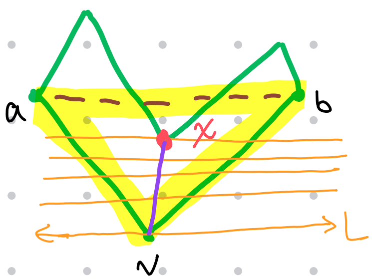
Let v be the lowest vertex of P (rightmost, if several).

least y-coordinate

Let a and b be the neighboring vertices of v.

If segment $\overline{ab}$ is a diagonal of P, then we are done.

Otherwise, triangle Δavb contains a vertex of P.

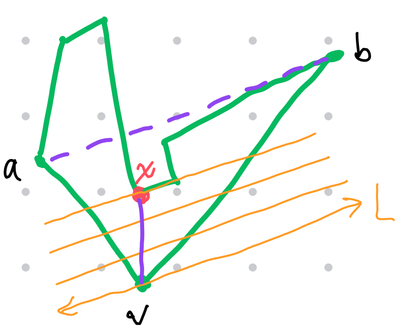


Let L be the line through v parallel to $\overline{ab}$.

Sweep L upward toward $\overline{ab}$, keeping its slope the same.

Let x be the first vertex that L meets along this sweep. (If L meets several vertices simultaneously, then choose one of them.)

Then $\overline{vx}$ is a diagonal of P. ■

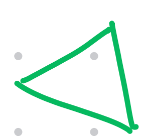


THEOREM 1.4: Every polygon has a triangulation

Proof: by induction on the number of vertices n.

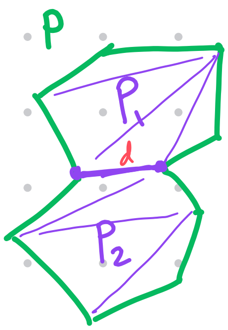
$n \geq 3$

BASE CASE: Suppose $n=3$. Then polygon P is a triangle, which is its own triangulation.



INDUCTION: Let $n > 3$. $\boxed{\text{Assume every polygon with fewer than } n \text{ vertices has a triangulation.}}$ inductive hypothesis

(We will show that every polygon with exactly n vertices has a triangulation.)



If P is a polygon with n vertices, then P has a diagonal d (by the lemma).

This diagonal cuts P into two polygons P_1 and P_2.

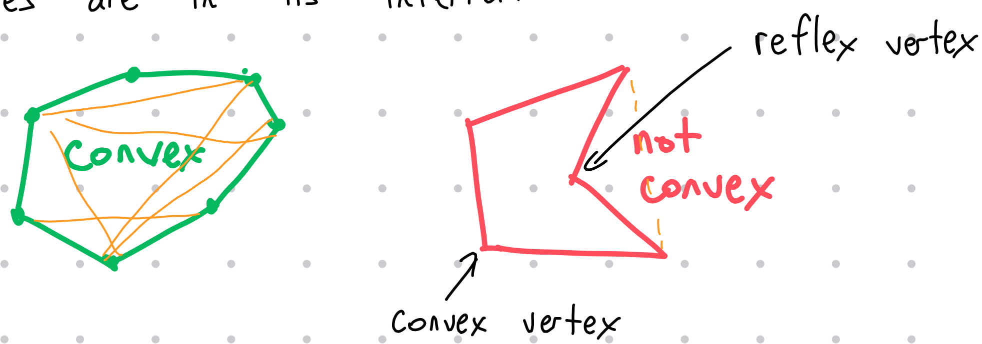
Then each of P_1 and P_2 has fewer than n vertices.

By the inductive hypothesis, both P_1 and P_2 have triangulations.

Combine the triangulations of P_1 and P_2, with the diagonal d, to obtain a triangulation of P. $\blacksquare$

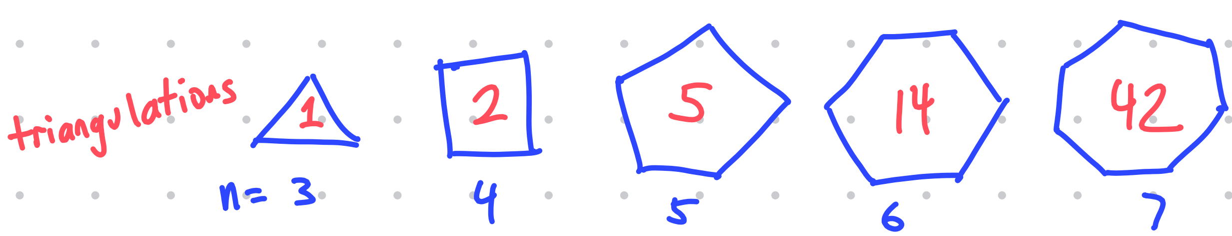
How many triangulations does an n-gon have?

A polygon is **convex** if all line segments between distinct vertices are in its interior.

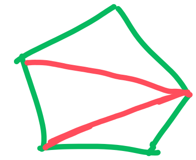


Among all n-gons, the convex n-gons have the most triangulations.

If P is a convex n-gon, then how many triangulations does P have?



n=5



3 triangles
2 diagonals

Polygon Triangulation Theorem

MATH 261 Computational Geometry

Complete the proof of the following theorem.

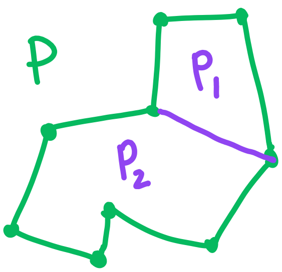
Theorem: Every triangulation of a polygon P with n vertices has $n-2$ triangles and $n-3$ diagonals.

Proof (by induction):
First state the base case. Explain why it is true.

If $n=3$, then P is a triangle.

Thus P is its own triangulation with $n-2=1$ triangle and $n-3=0$ diagonals.

Inductive hypothesis: Let $n > 3$ be an integer, and assume the statement is true for all polygons with fewer than n vertices.
Now explain how the statement follows for a polygon with n vertices.



Let P be a polygon with n vertices.

Choose a diagonal that cuts P into polygons P_1 and P_2 with n_1 and n_2 vertices, respectively.

Then $n_1 + n_2 = n + 2$. Certainly $n_1 < n$ and $n_2 < n$.

By the inductive hypothesis:

P_1 has $n_1 - 2$ triangles and $n_1 - 3$ diagonals.

P_2 has $n_2 - 2$ triangles and $n_2 - 3$ diagonals.

Then the number of triangles in P is:

$$(n_1 - 2) + (n_2 - 2) = n_1 + n_2 - 4 = n + 2 - 4 = n - 2,$$

and the number of diagonals in P is:

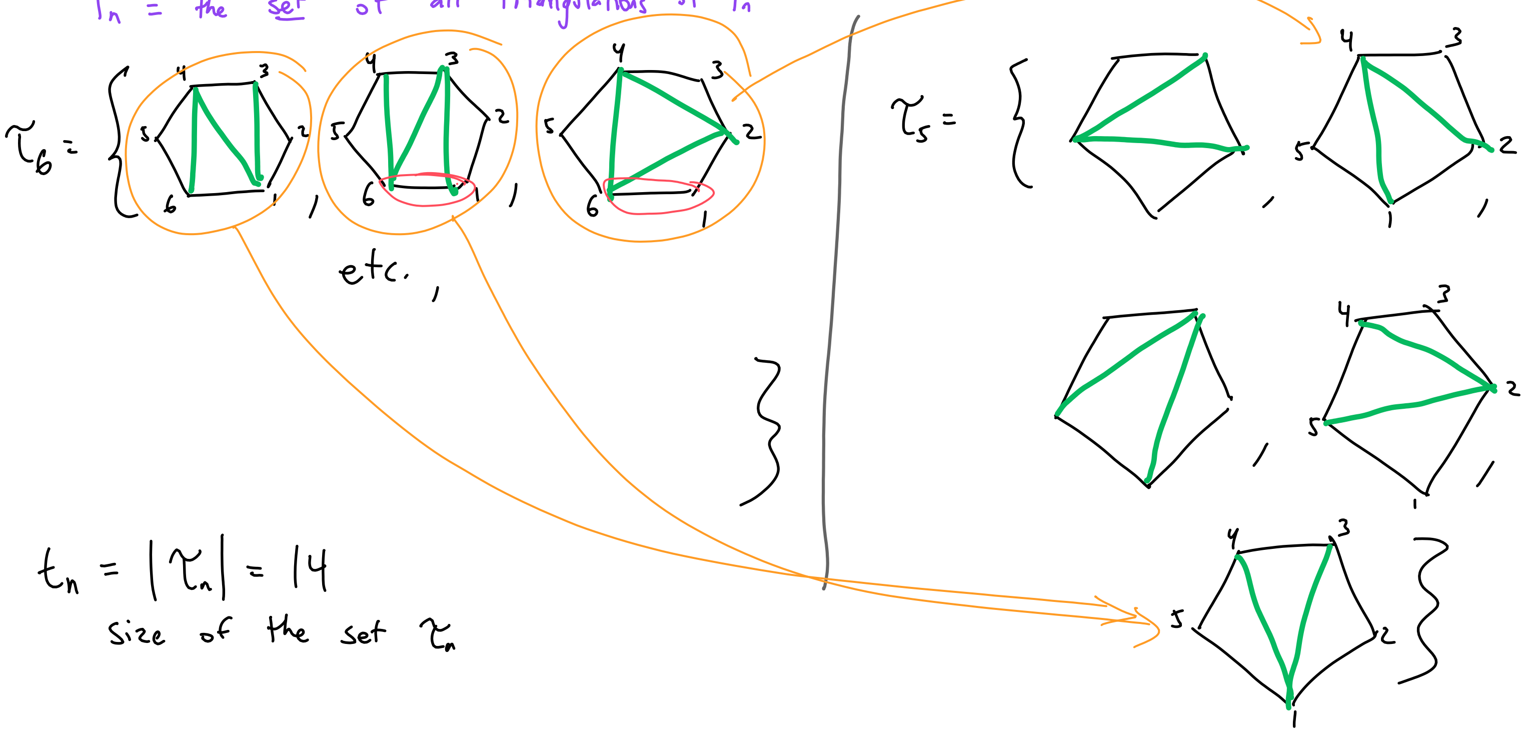
$$(n_1 - 3) + (n_2 - 3) + 1 = n_1 + n_2 - 5 = n + 2 - 5 = n - 3. \blacksquare$$

Edge Contraction Example

MATH 261 Computational Geometry

Let P_n be a convex polygon with vertices labeled 1 to n counterclockwise. Let $\mathcal{T}_n$ be the set of all triangulations of P_n, and let t_n be the number of elements of $\mathcal{T}_n$.
Define the map $\phi : \mathcal{T}_6 \to \mathcal{T}_5$ that contracts the edge $\{1, 6\}$ to the point 1. To illustrate this map, first draw all triangulations in $\mathcal{T}_6$ and $\mathcal{T}_5$. Then draw an arrow from each triangulation $T \in \mathcal{T}_6$ to $\phi(T)$.

$\mathcal{T}_n$ = the set of all triangulations of P_n



$t_n = |\mathcal{T}_n| = 14$

size of the set $\mathcal{T}_n$

We will return to this on Monday

Use your illustration to explain why

$$t_6 = \sum_{T \in \mathcal{T}_5} [\text{degree of vertex 1 in triangulation } T].$$

Then sum over all vertices of T to explain why

$$5 \cdot t_6 = 2 \cdot 7 \cdot t_5.$$

Now generalize. What equation relates t_{n+2} and t_{n+1}?